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A Preliminary View of Calculating Call Option Prices Utilizing Stochastic Volatility Models

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A Preliminary View of Calculating Call Option Prices Utilizing Stochastic Volatility Models

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Abstract

We will begin with a review of key financial topics and outline many of the crucial ideas utilized in the latter half of the paper. Formal notation for important variables will also be established. Then, a derivation of the Black-Scholes equation will lead to a discussion of its shortcomings, and the introduction of stochastic volatility models. Chapter 2 will focus on a variation of the CIR Model using stock price in the volatility driving process, and its behavior to a greater degree. The key area of discussion will be to approximate a hedging function for European call option prices by Taylor Expansion. We will apply this estimation to real data, and analyze the behavior of the price correction. Then make conclusions about whether stock price has any positive effects on the model.

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Introduction:

With the ever present need to provide a more accurate derivative pricing model, methods both old and new have to be tested in conjunction based on historical data. The biggest challenge is to find a model that holds with updated segments of historical data while maintaining a given set of parameters. Due to these complications, the projection model has to remain relatively uncomplicated and flexible in order to accommodate new information.

The Black Scholes Model has provided a fairly accurate account of derivative pricing for the last several decades. However, its accuracy is based upon occasions where the volatility remains relatively constant. The BS Model does not capture several key figures such as smile volatility, skew, and Kurtosis. Therefore we will explore these areas in order to present a tractable formula which can be used efficiently. The main focus is upon a variation of the Black Scholes that incorporates a stochastic volatility. As volatility clustering has just recently been introduced to simplify basic pricing and estimation problems by French mathematician *Jean-Pierre Fouque,* the main area of exploration will be to build upon some of his works. Later application of our model will be performed on historical values from indices such as S&P 500 and NASDAQ.

Background

Black Monday, October 19, 1987 was marked the largest one-day percentage decline in stock market history when stock markets around the world crashed. Beginning in Hong Kong and spreading through the rest of the world, such a high scale disaster carried an air of mystery as many could only conjecture as to the cause of such a fatal event. Many stock market hypothesis and assumptions of the economy were put into question but no definite conclusions were ever reached. It is peculiar that information surrounding this event seems to contradict the standard Black Scholes equation for option pricing[1] which follows the geometric Brownian

[1] $C + \frac{1}{2}\sigma^2 x^2 C_{xx} + r(xC_x - C) = 0$ where $X_t = x$

motion $\partial X_t = \mu X_t \partial t + \sigma(t, X_t) X_t \partial W_t$ where μ is drift, $W_t$is a Weiner process and $\sigma(t, X_t)$ is constant.

It is after this event that Stochastic Volatility models became popular for hedging and derivative pricing. However, any new model built on Black Scholes can only be changed in key places that can be refined. Therefore the main focus will be to change the volatility factor and to manipulate several assumptions. As Black Scholes is based upon historical data, this procedure will utilize the fact that the market is incomplete and will select a unique derivative pricing measure which reflects concerns of the economy from a sequence of measures.

Chapter 1: Stochastic Integrals

Wiener process

In order to model asset pricing on financial markets, calculations have to be made in continuous time. Therefore, stochastic differential equations have to be used in order to understand the theory behind market risk. In order to construct a Stochastic Integral, we will also introduce the diffusion process and define Brownian motion.

A stochastic Process X is diffusion if its local dynamics can be approximated by a stochastic difference equation of the type

$$X(t+\Delta t) - X(t) = \mu\big(t, X(t)\big)\Delta t + \sigma\big(t, X(t)\big)\Delta Z(t)$$

(1.1)

- $\mu\big(t, X(t)\big)$ is the drift term which determines the movement or velocity of the function
- $\sigma\big(t, X(t)\big)$ is the volatility or a disturbance factor
- $Z(t)$ is a Wiener Process

A stochastic process W is called a Wiener process given that:

1. W(0) = 0
2. W has independent increments. Let t_n be a time sequence with n=1, 2,… such that $0 < t_1 < \cdots < t_n$ the random variables ($W_{t_1},\ W_{t_2} - W_{t_1},\ W_{t_n} - W_{t_{n-1}}$) are all independent
3. For any $0 \le s < t$ the interval W(t) – W(s) is a mean-zero Gaussian distributed random variable with expectation zero and variance $E[(W_t - W_s)^2] = t - s$. W_t is N(0,t) -distributed

4. W almost surely has a continuous trajectory.

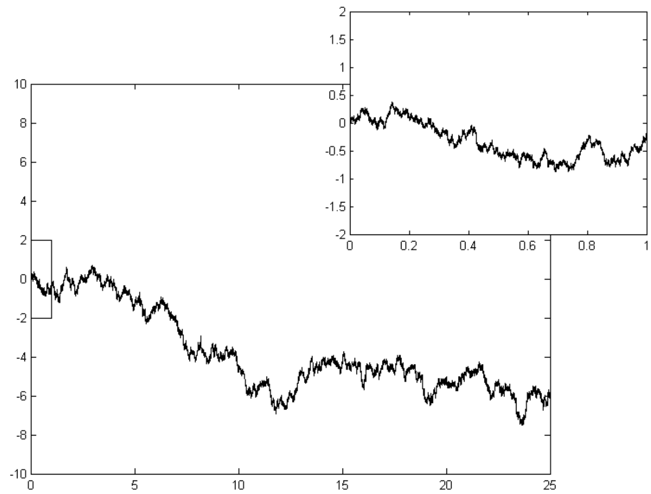

Figure 1-Wiener Process[2]

Martingales:

We will introduce the concept of martingales by loosely touching upon measure theory.

> **Definition:** Given a measurable space $(\Omega, \mathcal{F})$, a filtration is a sequence of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ such that $\mathcal{F}_t \subset \mathcal{F} \;\; \forall t \; in \; [0, \infty)$ and for each $n, m \in t$ such that $n \leq m$, we have that $\mathcal{F}_n \subseteq \mathcal{F}_m$.

We can describe a filtration $\mathcal{F}_t$ as information generated by a variable on the interval [0, t].

Proposition:

i. If X and Y are stochastic variables, such that X is $\mathcal{F}_t$ – measurable:

$$E[\, X \cdot Y|\mathcal{F}_t] = X \;\cdot E[\, Y|\mathcal{F}_t]$$

ii. For X a stochastic variable and $s, t \; in \; [0, \infty)$ such that s < t:

$$E[\, E[X|\mathcal{F}_t]|\mathcal{F}_s] = \; E[X|\mathcal{F}_s]$$

[2] A single realization of a one-dimensional Wiener process, Wikipedia

Proof:

i. The first proof is very straight forward. Since X is $\mathcal{F}_t$ – measurable, we know the exact value of X given the information $\mathcal{F}_t$. Therefore we can move X out of the expectation because it can be treated as a constant.

ii. The second proposition comes from the law of total expectation. However, intuitively we can say that if we take the expectation of X, we would integrate it over the entire probability space, but with a conditional expectation, we would only integrate over the filtration period $\mathcal{F}_t$.

It naturally follows that since s<t, if we integrate twice, over the larger filtration and then the smaller filtration periods, it would equal taking just the integral over the smaller filtration since the entire space $\mathcal{F}_s \subseteq \mathcal{F}_t$.

Definition: A stochastic process X is called an $\mathcal{F}_t$ – martingale if the following conditions hold:

- X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$
- $\forall t,\ E[|X(t)|] < \infty$
- $\forall s\ \&\ t\ with\ s \leq t, E[X(t)|\ \mathcal{F}_s] = X(s)$

Definition:

- a process that satisfies: $\forall s\ \&\ t\ with\ s \leq t$, $[X(t)|\ \mathcal{F}_s] \leq X(s)$ is called a Supermartingale
- If $[X(t)|\ \mathcal{F}_s] \geq X(s)$ then X is a Submartingale.

Proposition: For any process $g \in\ \mathcal{L}^2[s,t]$, the following holds,

$$E\left[\int_s^t g(u)\partial W(u)\middle|\mathcal{F}_s^w\right] = 0 = 0$$

(1.2)

It follows that for any process $g \in \mathcal{L}^2$

$$X(t) = \int_0^t g(s)\partial W(s)$$

(1.3)

is a $\mathcal{F}_s^w$ – martingale

Proof: Let $s, t \in [0, \infty)$ and s < t, then

$$E[X(t)|\mathcal{F}_s^w] = E\left[\int_0^t g(u)\partial W(u)\,\middle|\mathcal{F}_s^w\right]$$

$$= E\left[\int_0^s g(u)\partial W(u)\,\middle|\mathcal{F}_s^w\right] + E\left[\int_s^t g(u)\partial W(u)\,\middle|\mathcal{F}_s^w\right]$$

$$= \int_0^s g(u)\partial W(u) + 0 = X(s)$$

(1.4)

Assuming integrability, the inverse of the above proposition is shown below.

Martingale representation theorem: If a stochastic process X is a martingale with respect to the filtration generated by a Brownian motion, then

$$\partial X(t) = g(t)\partial W(t)$$

(1.5)

Stochastic Integrals

In order to define and construct stochastic integrals, we will use the conditions applied to $\mathcal{L}^2$ space, the space of Lebesgue integrable functions defined on $\mathbb{R}^2$.

Definition: Let $(X_t)_{0\le t<T}$ be a stochastic process adapted to the $(\mathcal{F}_t)_{(0\le t\le T)}$ filtration for finite T, we say that $(X_t)_{0\le t<T}$ is in $\mathcal{L}^2[0,T]$ if

$$\int_0^T E\left[{X_t}^2\right] \partial t < \infty$$

(1.6)

For any general process $f \in \mathcal{L}^2[0,T]$, there exists a sequence f_n of simple functions that converge to f so that $\forall \varepsilon > 0 \ and \ T \ge 0$

$$\int_0^T E[\{f_n - f\}^2] \, \partial s < \varepsilon$$

(1.7)

Then, if $\forall n, \int_a^b f_n(s)\partial W\,(s)$ is well defined and convergent to $\int_a^b f(s)\partial W(s) \ \ as \ n \to \infty$, we can define a stochastic integral

$$\int_a^b f(s)\,\partial W(s) = \lim_{n\to\infty} \sum_{i=1}^{n} f_n\left(W_{t_i} - W_{t_{i-1}}\right) = \lim_{n\to\infty} \int_a^b f_n(s)\partial W(s)$$

(1.8)

Proposition: If X_t is defined as above to be finite then the following relations hold:

$$E\left[\int_0^T X_t \partial W_t\right] = 0$$

$$E\left[\left(\int_0^T X_t\,\partial W_t\right)^2\right] = \int_0^T E\left[{X_t}^2\right] \partial t$$

$$\int_0^T X_t\,\partial W_t \ is \ \mathcal{F}_T \ measurable$$

(1.9)

We will create a partition $P = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = T\}$ where each t_i has equal length. We can then define a wiener process on this partition. Using the property of independent increments of Brownian motion, we have that

$$E\left[\sum_{i=1}^{n}\left(X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})\right)^2\right] = E\left[\sum_{i=1}^{n}\left(X_{t_{i-1}}\right)^2(t_i - t_{i-1})\right] \ for\ t \leq T$$

(1.10)

where the right side is equal to $E\left\{\int_0^T X_t{}^2\,\partial t\right\}$ which is finite. Thus if we take the limit as n approaches infinity, the partition becomes infinitesimal and we can approximate the stochastic integral of X_t with respect to the Brownian motion W_t in $\mathcal{L}^2(\Omega)$.

$$\lim_{n\to\infty}\sum_{i=1}^{n} X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) = \int_0^t X_s \partial W_s$$

(1.11)

Ito's Formula

Ito's formula was created to address the differentiation of stochastic processes. Normally, we can apply the chain rule to differentiation, however in the case of Wiener processes; these are not differentiable in the normal sense. Therefore, the Ito's formula is used to correct this discrepancy.

We will begin with the standard stochastic differential equation $\frac{\partial X_t}{X_t} = \mu\partial t + \sigma\partial W_t$. If we integrate both sides of the equation for $t \geq 0$ we get:

$$X_t = X_0 + \mu\int_0^t X_s\,\partial s + \sigma\int_0^t X_s \partial W_s$$

(1.12)

Where X_0 is a constant initial condition, assumed to be independent of the Brownian motion and square integrable. In the more general case, we will define $\mu(s, X_s) = \mu x$ and $\sigma(t, x) = \sigma x$; both independent of t, and x differentiable.

Ito's formula: If X has the stochastic differential $\partial X(t) = \mu(t, X_t)\partial t + \sigma(t, X_t)\partial W(t)$, with μ and σ as adapted processes, then the process $Z(t) = f(t, X_t)$ has a stochastic differential given by

$$\partial f(t, X_t) = \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right\}\partial t + \sigma\frac{\partial f}{\partial x}\partial W(t)$$

(1.13)

Where in the more general case, it can then be shown that the stochastic differential ∂f is equal to

$$\partial f\big(t, X(t)\big) = \frac{\partial f}{\partial t}\partial t + \frac{\partial f}{\partial x}\partial X + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\partial X)^2$$

(1.14)

with the conditions

$$(\partial t)^2 = 0,$$

$$\partial t\partial W = 0,$$

$$(\partial W)^2 = \partial t$$

(1.15)

Proof: We will begin by creating the partition $P = \{t_0 = 0 < t_1 < t_2 < \cdots < t_n = t\}$ with intervals of uniform length. Then integrate the differential $\partial f(W_t)$, this is equivalent to $f(W_t) - f(W_0)$, and apply Taylor's theorem. As each interval is of uniform length we have:

$$f(W_t) - f(W_0) = \sum_{i=1}^{n}\big(f(W_{t_i}) - f(W_{t-1})\big)$$

$$= \sum_{i=1}^{n} f'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) + \frac{1}{2}\sum_{i=1}^{n} f''(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2 + \cdots$$

As all the higher order terms converge to zero, we will only focus on the first two terms. As n approaches infinity, the Brownian motion intervals approaches zero.

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)\partial W_s + \frac{1}{2}\int_0^t f''(W_s)\partial s$$

(1.16)

The first equality comes directly from the stochastic integral of X_t. The second term is an application of the quadratic variation $\langle Y_t \rangle$ where $Y_t = \int_0^t X_s \partial W_s$.[3]

$$\langle Y_t \rangle = \lim_{n\to\infty} \sum_{i=1}^{n} (Y_{t_i} - Y_{t_{i-1}})^2 = \int_0^t X_s^2 \, \partial s$$

(1.17)

Once we differentiate both sides we arrive at the simple form:

$$\partial f(W_t) = f'(W_t)\partial W_t + \frac{1}{2} f''(W_t)\partial t$$

(1.18)

Now if we apply the more general formula dependent on t and X_t to the standard Brownian motion $\partial X_t = \mu(t, X_t)\partial t + \sigma(t, X_t)\partial W_t$, we get

$$\partial f(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu(t, X_t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}\right)\partial t + \sigma^2(t, X_t)\frac{\partial f}{\partial x}\partial W_t$$

(1.19)

[3] Note this is different from the notation for centering condition for Poisson process later in the paper

Multidimensional Ito's Formula

In the case where $f: \mathcal{R}_+ \times \mathcal{R}^n \to \mathcal{R}$ becomes a multidimensional continuous mapping, where $Z(t) = f(t, X(t))$, the dynamics for X becomes a vector process. Let $X = (X_1, X_2, \dots, X_n)$ and $W = (W_1, W_2, \dots, W_m)$

$$\partial X_i(t) = \mu_i(t)\partial t + \sum_{i=1}^{m} \sigma_{ij}(t)\partial W_j(t)$$

(1.20)

We will define μ, W, and σ as vector processes

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$$

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_m \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1d} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nd} \end{bmatrix}$$

Thus X-dynamics remains of the form

$$\partial X(t) = \mu(t)\partial t + \sigma(t)\partial W(t)$$

(1.21)

N dimensional Ito's formula: If X is an n-dimensional process with the dynamics given above, then the following hold

The process $f\big(t, X(t)\big)$ has a stochastic differential given by

$$\partial f\big(t, X(t)\big) = \left\{\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}\partial t + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\sigma_i \partial W$$

(1.22)

Where $\sigma_i = [\sigma_{i1}, \dots, \sigma_{im}]$ and $C = \sigma\sigma^T$

It can then be shown that the differential itself is given by

$$\partial f\big(t, X(t)\big) = \frac{\partial f}{\partial t}\partial t + \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\partial X_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}\partial X_i \partial X_j$$

(1.23)

where

$$(\partial t)^2 = 0,$$

$$\partial t \partial W = 0,$$

$$(\partial W_i)^2 = \partial t,$$

$$\partial W_i \partial W_j = 0 \; for \; i \neq j$$

Geometric Brownian Motion

Next we will introduce the standard building block for the rest of the section. The Geometric Brownian Motion is one of the two generalizations of the simplest ODE that is used in finance. This formula, in its simplest form was introduced earlier and consists of a standard wiener process with drift and velocity $\mu, and\ \sigma$.

$$\partial X_t = \mu X_t \partial t + \sigma X_t \partial W_t$$

(1.24)

Where X_0 is the initial price. We have more generally,

$$\partial X_t = \mu(t, x)\partial t + \sigma(t, x)\partial W_t$$

(1.25)

Infinitesimal Operator

We will follow the time-homogeneous process that solves the GBM model given above in the previous section. We will define the differential operator $\mathcal{L}$ acting on a twice differentiable function g such that

$$\mathcal{L}g(t,x) = \mu g'(t,x) + \frac{1}{2}\sigma^2(x)g''(t,x)$$

(1.26)

Similarly, in the case of a multidimensional equation, differential operator $\mathcal{L}$ of X is defined for a function $g(x) \in \mathcal{R}^n$ by

$$\mathcal{L}g(t,X_t) = \sum_{i=1}^{n} \mu_i(t,X_t)\frac{\partial g}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i\sigma_j\frac{\partial^2 g}{\partial x_i \partial x_j}$$

(1.27)

In terms of the Infinitesimal Operator $\mathcal{L}$, Ito's formula gives, in the single case

$$\partial g(X_t) = \mathcal{L}g(X_t)\partial t + g'(X_t)\sigma(X_t)\partial W_t$$

(1.28)

And in the multiple dimensional case we have

$$\partial g(t,X_t) = \left\{\frac{\partial g}{\partial t} + \mathcal{L}g\right\}\partial t + \nabla_x g(X_t)\sigma(X_t)\partial W_t$$

(1.29)

Where $\nabla_x g = \frac{\partial g}{\partial x_1} + \frac{\partial g}{\partial x_2} + \cdots + \frac{\partial g}{\partial x_n}$

Black-Scholes PDE

In 1973, Robert Merton published a paper that expanded the world of option pricing. Based on the work of Fischer Black and Myron Scholes, the Black-Scholes analysis of European options is able to calculate a unique trading strategy to replicate the payoff $h(X_t)$ of an option at maturity. This replicating strategy is a dynamic strategy that eliminates risk by creating a balanced portfolio where a loss on one portion is covered by a gain in another.

The Black-Scholes Model operates on the following assumptions:

- The price of the underlying stock follows a geometric Brownian motion with constant drift and velocity
- There is no arbitrage
- There are no taxes or transaction costs
- The underlying stock is not a dividend stock
- Borrowing/lending operates on a constant risk-free rate r
- Trading is continuous
- Short-selling is allowed
- Securities are perfectly divisible
- The price process for a derivative asset is of the form $\Pi(t, \chi) = \mathrm{F}\big(\mathrm{t}, \mathrm{X}(\mathrm{t})\big)$

We will begin with a given market with two assets who's dynamics are given by

$$\partial B(t) = r\, B(t) \partial t$$

$$\partial X_t = X_t \mu(t, X_t) \partial t + X_t \sigma(t, X_t) \partial W_t \tag{1.30}$$

Since we have assumed that there exists no arbitrage, we can require that

$$P(t, X_t) = \ a_t X_t + b_t e^{rt}\, for\ 0 \le t \le T \tag{1.31}$$

After a direct application of Ito's formula to both sides, we have

$$\left(\frac{\partial P}{\partial t} + \mu X_t \frac{\partial P}{\partial x} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2}\right)\partial t + \sigma X_t \frac{\partial P}{\partial x}\partial W_t = (a_t \mu X_t + b_t r e^{rt})\partial t + a_t \sigma X_t \tag{1.32}$$

where we equate like terms to calculate a_t and b_t. The first is accomplished by simply equating the ∂W_t terms and the second utilizes the initial definition

$$a_t = \frac{\partial P}{\partial x}$$

$$b_t = (P - a_t X_t)e^{-rt} \tag{1.33}$$

Equating the ∂t terms gives

$$r\left(P - X_t \frac{\partial P}{\partial x}\right) = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2}$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} + r X_t \frac{\partial P}{\partial x} - rP = 0 \tag{1.34}$$

When $P(t, x)$ is the solution of the Black-Scholes PDE, the equation above is satisfied for any stock price. We will pull $P(t, x)$ from the equation and denote the rest of the equation by $\mathcal{L}_{BS}$.

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x \frac{\partial}{\partial x} - .\right)$$

$$\mathcal{L}_{BS} P = 0 \tag{1.35}$$

Black-Scholes Equation: If we have a market with two assets as given in (1.30) and the no arbitrage assumptions in (1.31) then $P(t,x)$ is a pricing function that is a unique solution of the boundary value problem on the domain $[0,T] \times \mathcal{R}_+$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} + rX_t \frac{\partial P}{\partial x} - rP = 0$$

with the final condition that $P(T,x) = h(x)$.

The rate of return μ does not enter into the equation of this portfolio. For example, let (m_1, m_2) be two different portfolios, with returns (μ_1, μ_2) respectively [4]. However, as long as the historical volatility prevails, both portfolios will reach the same P.

Given a European derivative, we define a call option and a put option to be the solution of the Black-Scholes partial differential equation with the following final conditions

$$h_c(x) = (x - K)^+$$

$$h_p(x) = (K - x)^+$$

(1.36)

Black-Scholes Formula: given the above final conditions, the price of a European option at time t with asset price $X_t = x$ has a closed form solution

$$C(t,x) = x\Phi_{0,1}(d_1) - Ke^{-r(T-t)}\Phi_{0,1}(d_2)$$

(1.37)

$$P(t,x) = Ke^{-r(T-t)}\Phi_{0,1}(-d_2) - x\Phi_{0,1}(-d_1)$$

(1.38)

where $\Phi_{0,1}$ is the standard normal distribution

[4] $\mu_1 \neq \mu_2$

$$\Phi_{0,1}(\mathrm{z}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} \partial y$$

(1.39)

with d_1 and d_2 defined as:

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

(1.40)

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

(1.41)

The put-call parity relation is

$$C(t, X_t) - P(t, X_t) = X_t - Ke^{-r(T-t)}$$

(1.42)

Chapter 2

Volatility

One of the major problems with the Black Scholes model is that some of the assumptions are made to be idealistic and therefore, sometimes not compliant with real-world situations. Amongst several inconsistencies, one such example is the ability to hedge without transaction costs. Another is the assumption for constant volatility σ. When compared with historical volatility, σ fluctuations often resemble a smile shaped convex curve, rather than a straight line. This occurs more often in at-the-money options that have lower implied volatility rates than other options. We will discuss in this section, several approaches on using volatility.

Historic Volatility

The simplest and most intuitive way of using volatility is to use historical stock price data in order to estimate σ. We will begin with a standard Black-Scholes GBM and observe a discrete set S of stock prices at equidistant points with respect to t.

$$S = \{S_1, S_2, \dots, S_n\}$$

$$t = \{t_1, t_2, \dots, t_n\}$$

(2.1)

Let us assume that the stock prices follow a log normal distribution and define a new variable R for the rate of return, such that each R is independent and normally distributed.

$$R_i = \log\left(\frac{S_i}{S_{i-1}}\right)$$

(2.2)

Thus, with by general statistic formulas, we have the expected value and variance

$$E[R_i] = \left(\mu - \frac{1}{2}\sigma^2\right)(t_i - t_{i-1})$$

$$Var[R_i] = \sigma^2(t_i - t_{i-1})$$

(2.3)

We can estimate σ with the sample variance s

$$s = \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \mu)^2$$

(2.4)

and therefore estimate σ by

$$\tilde{\sigma} = \frac{R_i}{\sqrt{t_i - t_{i-1}}}$$

(2.5)

The standard deviation of $\tilde{\sigma}$ is then

$$std(\tilde{\sigma}) \approx \frac{\tilde{\sigma}}{\sqrt{2n}}$$

(2.6)

Implied Volatility

Implied volatility is the value of σ that the market has implicitly used in order to value a benchmark option. It is most frequently used to show the difference between the Black-Scholes prices for European call options and the market option Prices. In fact, given an observed European Call option $C^{observed}$, with Strike K and expiration date T, the implied volatility σ_I is calculated by matching the value of the price of the observed option with the Black-Scholes formula.

$$C(t, x; K, T; \sigma_I) = C^{observed}$$

(2.7)

In other words, we find the value of σ that will provide the current market price of a given stock. The process for asymptotic estimation discussed later in this paper also relies on this format of estimation for implied volatility as it is both simple and provides an accurate representation of realistic data. It is here that we begin to see the limitations of the standard Black-Scholes model. If the implied volatility was plotted as a function of the 'exercise' price, what should be a horizontal straight line is in reality a convex curve. Options that are in the extremities (either too far out of the money or deep in the money) are traded at higher implied volatilities than those whose prices are close to the strike price.

Stochastic Volatility Models

In order to capture the smile effect, two Brownian motions are constructed, one to map the fluctuations in the data, and the other to drive its movement. In this process, several features of volatility are maintained. That is, volatility will remain positive, and its behavior can be assumed to be a mean-reverting process. In particular, changes in stock price inversely affect the volatility of the stock, which partially accounts for a skewed distribution for stock prices. This is also known as the leverage effect, which agrees with empirical data. We begin with the standard GBM model in (1.25)

$$\partial X_t = \mu(t,x)\partial t + \sigma(t,x)\partial W_t$$

where $\sigma(t,x)_{t\geq 0}$ is the volatility process. In this case, volatility will be modeled to have an independent random component of in order to accurately represent a volatility process that is not completely correlated with the Wiener process W. In the following sections we will explore the Feller (CIR)[5] process for mean-reverting volatility that incorporates a second stochastic differential ∂Y_t of the form

$$\partial Y_t = \alpha(m - Y_t)\partial t + \beta \partial \hat{Z}_t \tag{2.8}$$

[5] Feller or Cox-Ingersoll-Ross Model

where $\hat{Z}_t$ is a Brownian Motion that is correlated with W_t. Z_t is independent of W_t

$$\hat{Z}_t = \rho W_t + \sqrt{1 - p^2} Z_t \tag{2.9}$$

We have that α is the rate of mean reversion and m is a long-term mean of Y, such that a drift term pulls Y towards the mean with respect to the long-run distribution of y.

Mean Reversion

Typically, a stock has both high and low prices but the principle of mean reversion refers to the idea that extremes for a stock price are temporary and will eventually 'revert' back to an average price. This process can be modeled by a linear coefficient in the drift term of the volatility process, or in the drift of an underlying process.

In the CIR Model, the coefficient for mean reversion is the first part of the volatility driving process

$$\partial Y_t = \alpha r_i (m - Y_t) \partial t + \cdots \partial \widetilde{Z}_t$$

We have here that m is the long term mean of Y and αr_i is the rate of mean reversion.

CIR Model

There are two major models for stochastic volatility, the Vasicek Model and the Cox-Ingersoll-Ross Model. The reason for their popularity is because they provide clean-cut closed form solutions for interest rate derivatives that are tractable. Therefore many variations of these models are available today.

Also known as the Feller Model, the CIR model we will be utilizing, follows the mean-reverting stochastic differential equation as the driving process.

$$\partial Y_t = \alpha r_i (m - Y_t) \partial t + \beta \sqrt{r_i} d\tilde{Z}_t$$

$$\tilde{Z}_t = \rho W_t + \sqrt{1 - p^2} Z_t$$

(2.10)

In this model, α, m, β are positive constants and r_i represents another process which introduces another dimension to the problem. For our case, we have that $r_i = x$ because we believe that having the stock price as a driving process of Y_t will provide more accuracy. $\tilde{Z}_t$ is a wiener process that is correlated to $W_t$but Z_t and W_t are independent. α is the rate of mean reversion and m is the long term mean of Y_t

CIR approximation:

Here we have the mean-reverting CIR process where W and Z are independent Brownian motions and ρ is the correlation between price and volatility shocks. $|\rho| < 1$ We will begin with a derivation of the derivation of the PDE given below in order to calculate the Pricing function of the CIR Model.

$$\partial X_t = \mu X_t \partial t + \sigma_t X_t \partial W_t$$

$$\sigma_t = e^y$$

$$\partial Y_t = \alpha r_i (m - Y_t) \partial t + \beta \sqrt{r_i} \partial \tilde{Z}_t$$

$$\tilde{Z}_t = \rho W_t + \sqrt{1 - p^2} Z_t$$

(2.11)

With application of multi-Dimensional Ito's formula on P, which is the price of a European derivative with expiration date T_1, we have

$$\partial P(t, X_t, Y_t) = \frac{\partial P}{\partial t} \partial t + \frac{\partial P}{\partial x} \partial X_t + \frac{\partial P}{\partial y} \partial Y_t + \frac{\partial^2 P}{\partial x dy} \partial X_t \partial Y_t + \frac{1}{2} \frac{\partial^2 P}{\partial x^2} \partial X^2{}_t + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \partial Y^2{}_t$$

(2.12)

$$= \frac{\partial P}{\partial t}\partial t + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}(e^{2y}x^2\partial t) + \frac{1}{2}\frac{\partial^2 P}{\partial x \partial y}(\mu x \partial t + e^y x \partial W_t)\big(\alpha x(m-y)\partial t + \beta\sqrt{x}\partial \tilde{Z}_t\big)$$

$$+ \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(\beta^2 x \partial t) + \frac{\partial P}{\partial x}(\partial X_t) + \frac{\partial P}{\partial y}(\partial Y_t)$$

$$= \left(\frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + \rho\beta e^y x^{3/2}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\beta^2 x \frac{\partial^2}{\partial y^2}\right)P\partial t + \frac{\partial P}{\partial x}\partial X_t + \frac{\partial P}{\partial y}\partial Y_t$$

(2.13)

We will make the assumption that the P can be hedged by a process (a_t, b_t, c_t) based upon the underlying stock, risk free asset, and a second derivative with expiration date T_2. There is equality because of the assumption that the market is arbitrage free.

$$P(T_1, X_t, Y_t) = a_{T_1}X_{T_1} + b_t\beta_{T_1} + c_{T_1}dP^{[2]}\big(T_1, X_{T_1}, Y_{T_1}\big)$$

$$= a_t X_t + b_t e^{rt} + c_t P^{[2]}(t, X_t, Y_t)$$

(2.14)

$P^{[2]}$ has the same payoff function as P however a different expiration date $T_2 > T_1 > t$. The portfolio is also self-financing so

$$\partial P(t, X_t, Y_t) = a_t \partial X_t + b_t r e^{rt} + c_t \partial P^{[2]}(t, X_t, Y_t)$$

(2.15)

We then apply the multi-Dimensional Ito formula to the right side of figure (2.14),

$$= b_t r e^{rt}\partial t + \left(a_t + c_t\frac{\partial P^{[2]}}{\partial y}\right)\partial X_t + c_t\frac{\partial P^{[2]}}{\partial t}\partial Y_t + c_t\frac{1}{2}\beta^2 x\frac{\partial^2 P^{[2]}}{\partial x^2}\,\partial t$$

$$+ c_t(\mu X_t \partial t + e^y x \partial W_t)(\alpha x(m-y)\partial t + \beta\sqrt{x}\partial\tilde{Z}_t)\frac{\partial^2 P^{[2]}}{\partial x \partial y}$$

$$= \left[b_t r e^{rt} + c_t\left(\frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\beta^2 x\frac{\partial^2}{\partial y^2} + \rho\beta e^y x^{3/2}\frac{\partial^2}{\partial x \partial y}\right)P^{[2]}\right]\partial t$$

$$+ \left(a_t + c_t\frac{\partial P^{[2]}}{\partial y}\right)\partial X_t + c_t\frac{\partial P^{[2]}}{\partial t}\partial Y_t$$

$$(2.16)$$

Thus we have that

$$\left(\frac{\partial}{\partial t}+\frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2}+\rho\beta e^y x^{3/2}\frac{\partial^2}{\partial x\partial y}+\frac{1}{2}\beta^2 x\frac{\partial^2}{\partial y^2}\right)Pdt+\frac{\partial P}{\partial x}\partial X_t+\frac{\partial P}{\partial y}\partial Y_t$$

$$=\left[b_t re^{rt}+c_t\left(\frac{\partial}{\partial t}+\frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2}+\frac{1}{2}\beta^2 x\frac{\partial^2}{\partial y^2}+\rho\beta e^y x^{3/2}\frac{\partial^2}{\partial x\partial y}\right)P^{[2]}\right]\partial t$$

$$+\left(a_t+c_t\frac{\partial P^{[2]}}{\partial y}\right)\partial X_t+c_t\frac{\partial P^{[2]}}{\partial t}\partial Y_t$$

$$(2.17)$$

The next step was to equate the terms from both sides to solve for the hedging process. We calculated a_t by equating the dX_t terms from equation (2.17)

$$\frac{\partial P}{\partial x}\partial X_t=\left(a_t+c_t\frac{\partial P^{[2]}}{\partial y}\right)\partial X_t$$

$$a_t=c_t\frac{\partial P^{[2]}}{\partial y}-\frac{\partial P}{\partial x}$$

$$(2.18)$$

c_t was calculated by equating the dY_t terms.

$$\frac{\partial P}{\partial y}\partial Y_t=c_t\frac{\partial P^{[2]}}{\partial t}\partial Y_t$$

$$c_t=\frac{\partial P/\partial y}{\partial P^{[2]}/\partial t}$$

$$(2.19)$$

Substitute for a_t and c_t in figure 2.14 at time t solves for b_t.

$$P(t,X_t,Y_t)=a_tX_t+b_te^{rt}+c_tP^{[2]}(t,X_t,Y_t)$$

$$b_t = \frac{(P - a_t X_t - c_t P^{[2]})}{e^{rt}}$$

(2.20)

By equating the ∂t terms from both sides of the equation (1.6) and substituting for the process (a_t, b_t, c_t) we can find a direct relation between P and $P^{[2]}$.

An abbreviation ω will be used from here on

$$\omega = \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\rho\beta e^y x^{3/2}\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\beta^2 x\frac{\partial^2}{\partial y^2}$$

(2.21)

$$\left(\frac{\partial}{\partial t} + \omega\right)P = \left(\frac{(P - a_t X_t - c_t P^{[2]})}{e^{rt}} re^{rt} + c_t\left(\frac{\partial}{\partial t} + \omega\right)P^{[2]}\right)$$

$$\left(\frac{\partial}{\partial t} + \omega + r\left(x\frac{\partial}{\partial x} - 1\right)\right)P = c_t\left(\frac{\partial}{\partial t} + \omega + r\left(x\frac{\partial}{\partial x} - 1\right)\right)P^{[2]}$$

Then, replace c_t with the equation in (2.19). $\left(\frac{d}{dt} + \omega + r\left(x\frac{d}{dx} - .\right)\right)$ is a Black-Scholes differential operator with volatility parameter e^y.[6]

$$\left(\frac{\partial P}{\partial y}\right)^{-1}\left(\frac{\partial}{\partial t} + \omega + r\left(x\frac{\partial}{\partial x} - 1\right)\right)P = \left(\frac{\partial P^{[2]}}{\partial y}\right)^{-1}\left(\frac{\partial}{\partial t} + \omega + r\left(x\frac{\partial}{\partial x} - 1\right)\right)P^{[2]}$$

(2.22)

As both sides of the equation depend on different expiration dates $T_1, and\ T_2$, we equate (2.21) and (2.22) to a function which does rely on an expiration date. This function is denoted by

$$\alpha x(m - y) - \beta\sqrt{x}\left(\rho\frac{\mu - r}{e^y} + \gamma(t, X_t, Y_t)\sqrt{1 - \rho^2}\right)$$

(2.23)

[6] In the equation The period . within the parenthesis is the indicator function with value 1.

Where $\gamma(t, X_t, Y_t)$ is an arbitrary function. Here we introduce $P^*(t, x, y)$ whose dependence on the expiration date is suppressed.

$$\left(\frac{\partial}{\partial t} + \omega + r\left(x\frac{\partial}{\partial x} - 1\right)\right)P^* + \left[\alpha x(m - y) - \beta\sqrt{x}\left(\rho\frac{\mu - r}{e^y} + \gamma(t, X_t, Y_t)\sqrt{1 - \rho^2}\right)\right]\frac{\partial P^*}{\partial y} = 0$$

(2.24)

where (2.25) is the Black-Scholes operator with volatility e^y

$$\frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - 1\right)$$

(2.25)

(2.26) is the correlation coefficient.

$$\rho\beta e^y x^{3/2}\frac{\partial^2}{\partial x\partial y}$$

(2.26)

(2.27) represents the infinitesimal generator of the OU process Y_t

$$\frac{1}{2}\beta^2 x\frac{\partial^2}{\partial y^2} + \alpha x(m - y)\frac{\partial}{\partial y}$$

(2.27)

Lastly, is the combined market price of volatility risk, or the premium

$$\beta\sqrt{x}\,\Lambda(t, x, y)\frac{\partial}{\partial y}$$

(2.28)

where $\Lambda(t, x, y) = \rho\frac{\mu - r}{e^y} + \gamma(t, X_t, Y_t)\sqrt{1 - \rho^2}$

We conclude the section by stating that $P^*(t,x,y)$ must satisfy the PDE in (2.24) with the terminal condition $P^*(T,x,y) = h(x)$ on the domain $(-\infty,\infty)$.

Asymptotic Approximation

In the previous section we calculated the pricing function $P^*(t,x,y)$. We will continue by using asymptotic approximation in order to estimate the pricing function and the

There are three parameters that affect the rate of mean reversion in Figure 2.3

1. The volatility defined by $e^{Y_t=y}$at time t
2. The correlation time of (Y_t) denoted by $\varepsilon = \frac{1}{\alpha}$ where $\varepsilon > 0$. This is used to model volatility clustering.
3. β^2 is used to model the variance of the distribution of Y. It controls the long term fluctuations in the volatility.

We will then take the previous Black Scholes PDE in (2.3) and split it into 3 separate parts in regard to the order of their $\frac{1}{\varepsilon}$ terms. We define $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ as

$$\mathcal{L}_0 = x\left[(m-y)\frac{\partial}{\partial y} + v^2\frac{\partial^2}{\partial y^2}\right]$$

$$\mathcal{L}_1 = \rho v\sqrt{2}e^y x^{3/2}\,\frac{\partial^2}{\partial x \partial y} - v\sqrt{2}\,x^{1/2}\Lambda(t,x,y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - 1\right)$$

(2.29)

where

- $\beta = \frac{v\sqrt{2}}{\sqrt{\varepsilon}} = \alpha v\sqrt{2}$

- $\mathcal{L}_2$is the Black-Scholes operator at the volatility level $f(y) = e^y$. $\mathcal{L}_2$ will also be denoted as $\mathcal{L}_{BS}\big(f(y)\big)$ which will be used later.
- $\mathcal{L}_1$ contains a mix of partial derivatives
- $\alpha\mathcal{L}_0$is the infinitesimal generator of the process Y divided by the stock price x.

The pricing PDE then becomes

$$\left(\frac{1}{\varepsilon}x\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P(t,X,Y) = 0 \tag{2.30}$$

We will make an assumption that P can be written in the form below, since its closed form is a sum of infinite length,

$$P(t,X,Y) = P_0(t,X,Y) + \sqrt{\varepsilon}P_1(t,X,Y) + \varepsilon P_2(t,X,Y) + \varepsilon\sqrt{\varepsilon}P_3(t,X,Y) + \ldots \tag{2.31}$$

P is a martingale by the Risk Neutral Valuation formula $P(t,X,Y) = E[C/F]$. There is a PDE that P satisfies but in order to produce a relatively simple formula, all consecutive terms will be dropped aside from the first 4. Thus, the infinite expansion can be grouped in orders of ε

$$\frac{1}{\varepsilon}x\mathcal{L}_0P_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0P_1 + \mathcal{L}_1P_0) + (\mathcal{L}_0P_2 + \mathcal{L}_1P_1 + \mathcal{L}_2P_0) + \sqrt{\varepsilon}(\mathcal{L}_0P_3 + \mathcal{L}_1P_2 + \mathcal{L}_2P_1) + \cdots = 0 \tag{2.32}$$

Separating terms

We begin by ordering the like terms by multiples of $1/\sqrt{\varepsilon}$ and then equating each term to 0. Starting with the first two divergent terms[7], we must have $\frac{1}{\varepsilon}x\mathcal{L}_0P_0 = 0$ in order for the entire expansion to equal zero. Since $\mathcal{L}_0$has derivatives with respect to y, P_0 must be a function of t

[7] Since $\varepsilon \to 0$

and x only. Both derivative terms become zero once we take the derivative of P_0 with respect to y.

$$\mathcal{L}_0 P_0 = x\left[(m-y)\frac{\partial P_0}{\partial y} + \beta^2 \frac{\partial^2 P_0}{\partial y^2}\right]$$

$$\frac{\partial P_0}{\partial y} = \frac{\partial^2 P_0}{\partial y^2} = 0$$

(2.33)

In order to eliminate the $1/\sqrt{\varepsilon}$ term we must have that

$$\frac{1}{\sqrt{\varepsilon}}(x\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) = 0$$

(2.34)

The second term $\mathcal{L}_1 P_0 = 0$ because $\mathcal{L}_1$is also a function with derivatives with respect to y and we already mentioned that P_0 is a function of t and x only. Similarly P_1 must also be a function of t and x only in order to satisfy the condition $\mathcal{L}_0 P_1 = 0$. Thus neither P_0 nor P_1 depend on the volatility y.

Poisson Equation on $\mathcal{L}_0$

The next term in the sequence to eliminate is of the order 1.

$$(x\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) = 0$$

(2.35)

From before, we have that $\mathcal{L}_1 P_1 = 0$ since $P_1 = P_1(t,x)$ so the first order term is reduced to

$$x\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$$

(2.36)

If we fix the variable x as a constant, and only view the equation as a function of y, the equation becomes of the form

$$\mathcal{L}_0\chi + g = (m - y)\frac{\partial P_2}{\partial y} + \beta^2\frac{\partial^2 P_2}{\partial y^2} + g(y) = 0$$

(2.37)

Which is known as the Poisson equation for $\chi(y)$ with respect to $\mathcal{L}_0$. This equation only has a solution if $g(y)$ follows a centering condition with respect to $\mathcal{L}_0$.

$$\langle g(y)\rangle = \int g(y)\Phi(y)\partial y = 0$$

(2.38)

where $\Phi(y)$ is the invariant or Gaussian distribution with standard deviation v and expected value m.

$$\Phi(y) = \frac{1}{v\sqrt{2\pi}}e^{-(y-m)^2/2v^2}$$

(2.39)

Thus we will apply the centering condition on $\mathcal{L}_2P_0$. Since P_0 does not depend on y, we can simply state that

$$\langle\mathcal{L}_2P_0\rangle = \int \mathcal{L}_2\Phi(y)P_0\partial y = \langle\mathcal{L}_2\rangle P_0 = 0$$

which, from (2.29), becomes

$$\langle\mathcal{L}_2\rangle = \mathcal{L}_{BS}(\bar{\sigma})$$

(2.40)

where $\bar{\sigma}^2 = \langle f^2\rangle$ is the effective volatility. Therefore the zero order term $P_0(t,x)$ is the solution to the Black-Scholes Equation

$$\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0$$

(2.41)

with the terminating condition $P_0(T,x) = h(x)$. Therefore we have that the term $P_0(t,x) = C(t,x)$ given by the Black-Scholes formula in (1.37).

$$P_0(t,x) = C(t,x) = x\Phi_{0,1}(d_1) - Ke^{-r(T-t)}\Phi_{0,1}(d_2)$$

(2.42)

with the centering condition satisfied, we now have that

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = P_0(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)$$

$$= P_0\left[\frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - 1\right)\right.$$
$$\left. - \int \left[\frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - 1\right)\right]\frac{1}{v\sqrt{2\pi}}e^{-(y-m)^2/2v^2}\,\partial y\right]$$

(2.43)

Since the integral of the Gaussian Distribution from $(-\infty,\infty)$ is 1, the terms of $\langle \mathcal{L}_2 \rangle$ that do not depend on y cancel out the terms from $\mathcal{L}_2$

$$= \left[\frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} - \int\left[\frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2}\right]\frac{1}{v\sqrt{2\pi}}e^{-(y-m)^2/2v^2}\,\partial y\right]P_0$$

$$= \frac{1}{2}[e^{2y} - \bar{\sigma}^2\,]x^2\frac{\partial^2 P_0}{\partial x^2}$$

(2.44)

Therefore, since $x\mathcal{L}_0 P_2$ must equal $\mathcal{L}_2 P_0$, we must have that

$$P_2 = -\frac{1}{2}\mathcal{L}_0^{-1}(e^{2y} - \bar{\sigma}^2)x^3\frac{\partial^2 P_0}{\partial x^2}$$

$$= -\frac{1}{2}\big(\phi(y) + c(t,x)\big)x^3 \frac{\partial^2 P_0}{\partial x^2}$$

(2.45)

where c is a constant and $\phi(y)$ is the solution of another Poisson equation

$$\mathcal{L}_0\phi(y) = f(y)^2 - \langle f^2 \rangle$$

(2.46)

$\sqrt{\varepsilon}$ term

Having solved for $P_0(t,x)$ and $P_2(t,x)$, we move on to the $\sqrt{\varepsilon}$ order term in (2.32) which must equal 0 which gives us another Poisson equation with respect to $\mathcal{L}_0$

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

where $g = \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1$. We apply the centering condition to get the equation for $\langle \mathcal{L}_2 P_1 \rangle$

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = \langle \mathcal{L}_1 P_2 \rangle + \langle \mathcal{L}_2 P_1 \rangle = 0$$

$$\langle \mathcal{L}_1 P_2 \rangle = \mathcal{L}_1 \left[-\frac{1}{2}\phi(y)x^3 \frac{\partial^2 P_0}{\partial x^2} \right]$$

$$\langle \mathcal{L}_2 P_1 \rangle = \langle \mathcal{L}_2 \rangle P_1 = \langle \mathcal{L}_{BS} \rangle P_1 = -\langle \mathcal{L}_1 P_2 \rangle$$

$$= \frac{1}{2}\langle \mathcal{L}_1 \phi(y) \rangle x^3 \frac{\partial^2 P_0}{\partial x^2}$$

(2.47)

As $\mathcal{L}_1$is dependent on y, we will compute the centering condition on $\mathcal{L}_1\phi(y)$.

$$\mathcal{L}_1\phi(y) = \ \rho v\sqrt{2}\langle e^y \phi'(y) \rangle x^{3/2} \ \frac{\partial}{\partial x} - v\sqrt{2}\, x^{1/2} \langle \Lambda(t,x,y)\phi'(y) \rangle$$

(2.48)

Finally we conclude that

$$\mathcal{L}_{BS}(\bar{\sigma})P_1 = \frac{\sqrt{2}}{2}\,\rho v\langle e^y\phi'\rangle x^{9/2}\frac{\partial^3 P_0}{\partial x^3} + \left(\sqrt{2}\,\rho v\langle e^y\phi'\rangle - \frac{\sqrt{2}}{2}v\,\langle\Lambda\phi'\rangle\right)\mathrm{x}^{7/2}\frac{\partial^2 P_0}{\partial x^2}$$

(2.49)

where $P_1(T, x) = 0$

We will now modify P_1 slightly and introduce a new variable in order to produce some cleaner results.

$$\tilde{P}_1(t, x) = \sqrt{\varepsilon}P_1(t, x)$$

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_1 = H(t, x)$$

(2.50)

such that, the source term H and the small coefficients are given by:

$$H(t, x) = V_3 x^{9/2}\frac{\partial^3 P_0}{\partial x^3} + V_2\mathrm{x}^{7/2}\frac{\partial^2 P_0}{\partial x^2}$$

$$V_3 = \frac{\rho v}{\sqrt{2\alpha}}\langle e^y\phi'\rangle$$

$$V_2 = \frac{v}{\sqrt{2\alpha}}(2\,\rho\langle e^y\phi'\rangle - \langle\Lambda\phi'\rangle)$$

(2.51)

We will now introduce the identity

$$\mathcal{L}_{BS}(\bar{\sigma})(-(T - t)H) = H - (T - t)\mathcal{L}_{BS}(\bar{\sigma})H$$

(2.52)

With this result it can be shown that the correction for the $\sqrt{\varepsilon}$ term satisfies the Black-Scholes equation with a zero terminal condition given above. This is given by

$$\tilde{P}_1(t,x) = -(T-t)\left(V_3 x^{9/2}\frac{\partial^3 P_0}{\partial x^3} + V_2 \mathrm{x}^{7/2}\frac{\partial^2 P_0}{\partial x^2}\right)$$

(2.53)

We can deduce that the last term is zero since we have the equality

$$\mathcal{L}_{BS}(\bar{\sigma})\left(x^m \frac{\partial^n P_0}{\partial x^n}\right) = x^m \frac{\partial^n}{\partial x^n}\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0$$

(2.54)

Therefore we have that the corrected price is given by

$$P_0 + \tilde{P}_1(t,x)$$

(2.55)

Where P_0 is the Black-Scholes price with constant volatility $\bar{\sigma}$.

Put-Call Parity

We will now show that Put-Call Parity is preserved with the small correction that is introduced in the last section. As we know that put-call parity defines a relation between the price of a call option and a put option with identical strike price and Expiration date. We have that:

$$C_0(t,x) + KB(t,T) = P_0(t,x) - x$$

$$C_0(t,x) - P_0(t,x) = x - Ke^{-r(T-t)}$$

(2.56)

where $B(t,T)$ is a price of a bond with expiration date T and $B(t,T) = e^{-r(T-t)}$ at time t.

Similarly we can apply this to the corrections $\tilde{C}_1(t,x)$ and $\tilde{P}_1(t,x)$ and we have that Put-Call Parity is preserved:

$$C_0(t,x)+\tilde{C}_1(t,x)-\left(P_0(t,x)+\tilde{P}_1(t,x)\right)=x-Ke^{-r(T-t)}$$

(2.57)

Since we have the relation that,

$$\tilde{C}_1(t,x)-\tilde{P}_1(t,x)=-(T-t)\left(V_2x^{9/2}\frac{\partial^3 P_0}{\partial x^3}+V_3\mathrm{x}^{7/2}\frac{\partial^2 P_0}{\partial x^2}\right)(C_0-P_0)$$

$$=-(T-t)\left(V_2x^{9/2}\frac{\partial^3 P_0}{\partial x^3}+V_3\mathrm{x}^{7/2}\frac{\partial^2 P_0}{\partial x^2}\right)\left(x-Ke^{-r(T-t)}\right)$$

$$=0$$

(2.58)

Derivation of the Greek terms

We will now compute the corrected price $P_0+\tilde{P}_1(t,x)$ that was calculated earlier in (2.55). Using Call options, we have from (1.37) that C_{BS} is equal to the leading term P_0

$$C_{BS}(t,x)=x\Phi_{0,1}(d_1)-Ke^{-r(T-t)}\Phi_{0,1}(d_2)$$

$$\Phi_{0,1}(\mathrm{z})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-y^2/2}\partial y$$

$$d_{1,2}=\frac{\log\left(\frac{x}{K}\right)+\left(r\pm\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

From here, we will derive the first, second and third order derivative of C_{BS} with respect to x. Let us first introduce several relations between $d_1$and d_2

$$e^{-d_2^2/2}=\left(\frac{xe^{r(T-t)}}{k}\right)e^{-d_1^2/2}$$

$$\Phi'_{0,1}(\mathrm{z})=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

$$\frac{\partial d_{1,2}}{\partial x} = \frac{1}{x\bar{\sigma}\sqrt{T-t}}$$

(2.59)

In the case of Delta, we have from the chain rule and change of variables:

$$\frac{\partial P_0}{\partial x} = \Phi_{0,1}(d_1) + \frac{e^{-d_1^2/2}}{x\bar{\sigma}\sqrt{2\pi(T-t)}} - Ke^{-r(T-t)}\frac{e^{-d_2^2/2}}{x\bar{\sigma}\sqrt{2\pi(T-t)}}$$

$$= \Phi_{0,1}(d_1) + \frac{e^{-d_1^2/2}}{x\bar{\sigma}\sqrt{2\pi(T-t)}} - Ke^{-r(T-t)}\frac{\left(\frac{xe^{r(T-t)}}{k}\right)e^{-d_1^2/2}}{x\bar{\sigma}\sqrt{2\pi(T-t)}}$$

$$= \Phi_{0,1}(d_1)$$

(2.60)

Then we compute Gamma,

$$\frac{\partial^2 P_0}{\partial x^2} = d_1' \Phi_{0,1}'(d_1) = \left(\frac{e^{-d_1^2/2}}{x\bar{\sigma}\sqrt{2\pi(T-t)}}\right)$$

(2.61)

Lastly, we will calculate the third order derivative which is needed in order to compute $H(t,x)$. By the chain rule, we have

$$\frac{\partial^3 P_0}{\partial x^3} = \left(-\frac{1}{x^2}\right)\frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} + \left(\frac{1}{x\sigma\sqrt{2\pi(T-t)}}\right)\frac{e^{-d_1^2/2}}{x\bar{\sigma}\sqrt{T-t}}$$

$$= \frac{-e^{-d_1^2/2}}{x^2\bar{\sigma}\sqrt{2\pi(T-t)}}\left(1 + \frac{d_1}{\bar{\sigma}\sqrt{T-t}}\right)$$

(2.62)

We will now use the results of (2.61) and (2.62) to calculate H from (2.51)

$$H(t,x) = V_3 x^{9/2}\frac{\partial^3 P_0}{\partial x^3} + V_2 \mathrm{x}^{7/2}\frac{\partial^2 P_0}{\partial x^2}$$

$$V_3 x^{9/2}\left(\frac{-e^{-d_1^2/2}}{x^2\bar{\sigma}\sqrt{2\pi(T-t)}}\left(1+\frac{d_1}{\bar{\sigma}\sqrt{T-t}}\right)\right) + V_2 \mathrm{x}^{7/2}\left(\frac{e^{-d_1^2/2}}{x\bar{\sigma}\sqrt{2\pi(T-t)}}\right)$$

$$= \frac{x^{5/2}e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi(T-t)}}\left(V_2 - V_3\left(1+\frac{d_1}{\bar{\sigma}\sqrt{T-t}}\right)\right)$$

(2.63)

The small correction $\tilde{P}_1(t,x)$ is then given by

$$\tilde{P}_1(t,x) = -(T-t)H(t,x) = \frac{x^{5/2}e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi}}\left(V_3\frac{d_1}{\bar{\sigma}} + (V_3 - V_2)\sqrt{T-t}\right)$$

(2.64)

Implied volatilities

We will now use implied volatilities to estimate our two small parameters V_2 and V_3. Let us recall the implied volatility σ_I given in (2.7)

$$C_{BS}(t,x;K,T;\ \sigma_i) = C^{observed}(K,T)$$

We can expand σ_i with a Taylor series on the left $\sigma_I = \bar{\sigma} + \sqrt{\varepsilon}\sigma_1 + \cdots$ Then match the right side term-by-term with our approximated price

$$C_{BS}(t,x;K,T;\ \sigma_I) + \sqrt{\varepsilon}\sigma_1\frac{\partial C_{BS}}{\partial\sigma} + \cdots = P_0(t,x) + \tilde{P}_1(t,x)$$

(2.65)

$$C_{BS}(t,x;K,T;\ \sigma_i) = P_0(t,x)$$

$$\sqrt{\varepsilon}\sigma_1 = \tilde{P}_1(t,x)\left(\frac{\partial C_{BS}}{\partial \sigma}\right)^{-1}$$

(2.66)

Then, up to an error of $\mathrm{O}(\varepsilon)$, the implied volatility is given by

$$\sigma_I = \bar{\sigma} + \tilde{P}_1(t,x)\left(\frac{\partial C_{BS}}{\partial \sigma}\right)^{-1} + \mathrm{O}(\varepsilon)$$

(2.67)

Where the derivative of C_{BS} with respect to the volatility parameter is

$$\frac{\partial C_{BS}}{\partial \sigma} = \frac{x e^{-d_1^2/2}\sqrt{T-t}}{\sqrt{2\pi}}$$

(2.68)

When combined with the correction in (2.64), this yields

$$\sigma_i = \bar{\sigma} + \frac{x^{3/2} V_3 d_1}{\bar{\sigma}^2 \sqrt{T-t}} + \frac{x^{3/2}(V_3 - V_2)}{\bar{\sigma}} + \mathrm{O}(\varepsilon)$$

(2.69)

$$= \bar{\sigma} + \frac{x^{3/2} V_3}{\bar{\sigma}^3}\left(r + \frac{1}{2}\bar{\sigma}^2\right) - \frac{x^{3/2} V_2}{\bar{\sigma}} - \frac{x^{3/2} V_3}{\bar{\sigma}^3}\left(\frac{Log\left(\frac{K}{x}\right)}{T-t}\right) + \mathrm{O}(\varepsilon)$$

(2.70)

which is an affine function of the log-moneyness-to-maturity ratio up to order $\mathrm{O}(\varepsilon)$. Thus we have the LMMR as:

$$\sigma_I = a\left[\frac{Log\left(\frac{k}{x}\right)}{T-t}\right] + b + \mathrm{O}(\varepsilon)$$

$$a = -\frac{x^{3/2}V_3}{\bar{\sigma}^3}$$

$$b = \bar{\sigma} + \frac{x^{3/2}V_3}{\bar{\sigma}^3}\left(r + \frac{1}{2}\bar{\sigma}^2\right) - \frac{x^{3/2}V_2}{\bar{\sigma}}$$

(2.71)

Once we estimate a and b from data, the group parameters V_2 and V_3 are then given by

$$V_2 = \bar{\sigma}x^{-3/2}\left(\bar{\sigma} - b - a\left(r + \frac{1}{2}\bar{\sigma}^2\right)\right)$$

$$V_3 = -a\bar{\sigma}^3 x^{-3/2}$$

(2.72)

Data Results

In order to calculate the LMMR, we will test the accuracy of the estimates and the behavior of its volatility on real data. The contracts used were at least two months from expiration and were within 3% of the money to provide sufficient liquidity. The trading days used were 4/1/2004 to 6/29/2004 and 6 options with sufficient liquidity were chosen

Stock	Option	Least Squares	Standard Dev.	Data points
ACDO	DZU-HI			63
	a=	0.3444	0.0337	
	b=	0.1603		
ATK	ATK-HN			61
	a=	0.1177	0.0209	
	b=	0.1857		
BKST	BEQ-HD			53
	a=	-0.2870	0.0208	
	b=	0.0846		
AEP	AEP-HG			56
	a=	-0.1128	0.0170	
	b=	0.1222		
CA	CA-HY			56
	a=	0.2866	0.0245	
	b=	0.1879		
CYN	CYN-HN			58
	a=	0.5038	0.0318	
	b=	0.1631		

Figure 2- LMMR Calculations

A full explanation of how the data was calculated is available in the Appendix, along with scatter plots for each option. The daily prices for both options are also available.

Next, both historical and the effective volatility were calculated in order to solve for V_2, V_3 and $\tilde{P}_1$. Based on the CIR model, $f(y) = e^y$, we have that the effective volatility $\bar{\sigma}^2 = e^2$. A complete derivation is available in the appendix. Historical volatility was used in equation (2.73), along with the calculations above, to solve for V_2 and V_3. The effective volatility was used to calculate the option price modifier $\tilde{P}_1$ in equation (2.64).

Stock	Option			
ACDO	DZU-HI			
		V2	0.0870834	$x^{-3/2}$
		V3	-0.0235547	$x^{-3/2}$
ATK	ATK-HN			
		V2	0.6158615	$x^{-3/2}$
		V3	-0.0893741	$x^{-3/2}$
BKST	BEQ-HD			
		V2	0.0341941	$x^{-3/2}$
		V3	0.0032338	$x^{-3/2}$
AEP	AEP-HG			
		V2	0.2093617	$x^{-3/2}$
		V3	0.0152406	$x^{-3/2}$
CA	CA-HY			
		V2	0.0556682	$x^{-3/2}$
		V3	-0.0139617	$x^{-3/2}$
CYN	CYN-HN			
		V2	0.4608161	$x^{-3/2}$
		V3	-0.3556294	$x^{-3/2}$

Figure 3- V_2, V_3 Calculations

For simplicity, V_2 and V_3 were calculated without the daily returns. This value was then substituted into (2.64). With $V_2$and V_3, we solved for $\tilde{P}_1$ in equation (2.64). The modified prices are included in the appendix.

Conclusions

The CIR Model for option prices is a variation of the Ornstein-Uhlenbeck (OU) Model. We chose this model because although it is very complicated, it allows for flexibility and the freedom to explore from a learning perspective. We decided to use a variation of this model and test whether introducing the stock price in the volatility driving process would make the equation more sensitive and accurate. Once we solved for $\tilde{P}_1$, we observed that the correction was reasonable and surprisingly, equivalent to the pricing process of the OU model. The calibration of the variable $r_i = x$ which we had introduced provides the same error $\tilde{P}_1$ as the OU Model where $r_i = 1$.

Effects of different r_i

We then analyzed $\tilde{P}_1$ by re-calibrating r_i and re-applying the same steps to calculate $\tilde{P}_1$. In order to calculate the effects r_i have on the correction price, we focused mainly on specific changes in the formula caused by changing the powers of x. Thus we will outline the changes in our equation to the general case r_i

- In equation (2.29):
  - The multiplier for $\mathcal{L}_0$ is r_i
  - The x values in $\mathcal{L}_1$ are $x\sqrt{r_i}$ and $\sqrt{r_i}$ respectively
- Due to changes in the multiplier for $\mathcal{L}_0$, the x value in (2.45) for P_2 is $x^2 r_i$
- The x values in (2.49) for $\mathcal{L}_{BS}(\bar{\sigma})P_1$ are $x^3 r_i^{3/2}$ and $x^2 r_i^{3/2}$ respectively
- This causes the x values in the correction $\tilde{P}_1$ to change to $x r_i^{3/2}$ in equation (2.64)
- Lastly, the multipliers for V_2 and V_3 become $r_i^{-3/2}$ in equation (2.72)

Here we listed the powers of x from zero to two in the table below.[8]

[8] The first row, with x to the power 0 is the OU Model result while the x to the power 1 is our result. Columns two and three represent the equation for $\tilde{P}_1$ in (2.64). They were separated to show the differences in the exponential on the outside and the values of V_2 and V_3 inside the parentheses.

$r_i = x^\wedge$			P1
0	2.38711182	-0.0824098	-0.1967215
0.1	4.21434713	-0.046679	-0.1967215
0.2	7.44025543	-0.0264401	-0.1967215
0.3	13.135463	-0.0149764	-0.1967215
0.4	23.1901163	-0.008483	-0.1967215
0.5	40.9411906	-0.004805	-0.1967215
0.6	72.2799776	-0.0027217	-0.1967215
0.7	127.607309	-0.0015416	-0.1967215
0.8	225.285423	-0.0008732	-0.1967215
0.9	397.73209	-0.0004946	-0.1967215
1	702.179543	-0.0002802	-0.1967215
1.1	1239.66892	-0.0001587	-0.1967215
1.2	2188.58418	-8.989E-05	-0.1967215
1.3	3863.85478	-5.091E-05	-0.1967215
1.4	6821.4757	-2.884E-05	-0.1967215
1.5	12043.033	-1.633E-05	-0.1967215
1.6	21261.4764	-9.252E-06	-0.1967215
1.7	37536.2566	-5.241E-06	-0.1967215
1.8	66268.7075	-2.969E-06	-0.1967215
1.9	116994.66	-1.681E-06	-0.1967215
2	206549.231	-9.524E-07	-0.1967215

Figure 4-R values

Since this result is fairly surprising, we went back and reviewed the changes made to the equation. If we pay attention to equations (2.64) and (2.72) which becomes

$$\tilde{P}_1(t,x) = -(T-t)H(t,x) = \frac{x r_i^{3/2} e^{-d_1^2/2}}{\bar{\sigma}\sqrt{2\pi}}\left(V_3 \frac{d_1}{\bar{\sigma}} + (V_3 - V_2)\sqrt{T-t}\right)$$

$$V_2 = \bar{\sigma} r_i^{-3/2}\left(\bar{\sigma} - b - a\left(r + \frac{1}{2}\bar{\sigma}^2\right)\right)$$

$$V_3 = -a\bar{\sigma}^3 r_i^{-3/2}$$

(2.73)

We see that the r_i coefficients in V_2, V_3 and $\tilde{P}_1(t,x)$ cancel each other out. With this result, we conclude that the stock price has no effect on the volatility driving formula ∂Y_t in the CIR Model, as all powers of x lead to the same result for the small correction price $\tilde{P}_1$.

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Appendix

Data collecting

After obtaining 4 years of daily option prices, we are able to apply the formulas to real data. However, with such a large data set, there are many problems pulling together the relevant data needed to calculate the LMMR (log-moneyness-to-maturity-ratio). The data was separated into folders for months, with a data file for each day, containing over 150,000 rows of stock option data.

Each file was encoded as a .csv file (comma separated variables) and after exploring different methods, such as Matlab and Database organizing software, excel was chosen because it was the most convenient software for analyzing such large data sets.

A1 | f_x | A

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	A	22.07	*	A	DD	call	4/15/2005	4/1/2005 16:00	20	2.5	2.1	2.2	15	105	0.393106	0.912401	9.598019	0	0
2	A	22.07	*	A	PD	put	4/15/2005	4/1/2005 16:00	20	0.05	0	0.1	0	90	0.362779	-0.07174	8.94086	0	0
3	A	22.07	*	A	DX	call	4/15/2005	4/1/2005 16:00	22.5	0.5	0.3	0.35	5	1908	0.29831	0.381967	30.30621	0	0
4	A	22.07	*	A	PX	put	4/15/2005	4/1/2005 16:00	22.5	0.65	0.7	0.8	18	1842	0.302888	-0.61597	29.89626	0	0
5	A	22.07	*	A	DE	call	4/15/2005	4/1/2005 16:00	25	0.05	0.05	0.1	50	1752	0.461961	0.086679	8.103298	0	0
6	A	22.07	*	A	PE	put	4/15/2005	4/1/2005 16:00	25	2.85	2.95	3	0	743	0.436021	-0.92614	7.607423	0	0
7	A	22.07	*	A	DF	call	4/15/2005	4/1/2005 16:00	30	0.05	0	0.05	0	100	0.767518	0.021923	1.61635	0	0
8	A	22.07	*	A	PF	put	4/15/2005	4/1/2005 16:00	30	5.6	7.9	8	0	29	0.813294	-0.97086	1.935953	0	0
9	A	22.07	*	A	DG	call	4/15/2005	4/1/2005 16:00	35	0	0	0.05	0	0	1.077272	0.016421	0.900682	0	0
10	A	22.07	*	A	PG	put	4/15/2005	4/1/2005 16:00	35	10.3	12.9	13	0	3	1.147462	-0.97673	1.136889	0	0
11	A	22.07	*	A	EW	call	5/20/2005	4/1/2005 16:00	17.5	4.9	4.7	4.8	0	277	0.481988	0.922468	3.750478	0	0
12	A	22.07	*	A	QW	put	5/20/2005	4/1/2005 16:00	17.5	0.1	0.1	0.15	40	520	0.464059	-0.07076	3.632685	0	0
13	A	22.07	*	A	ED	call	5/20/2005	4/1/2005 16:00	20	2.75	2.45	2.55	10	238	0.377656	0.788091	9.552668	0	0
14	A	22.07	*	A	QD	put	5/20/2005	4/1/2005 16:00	20	0.3	0.3	0.4	2	1509	0.360108	-0.20264	9.756493	0	0
15	A	22.07	*	A	EX	call	5/20/2005	4/1/2005 16:00	22.5	0.95	0.8	0.9	72	5750	0.319291	0.463792	15.49352	0	0
16	A	22.07	*	A	QX	put	5/20/2005	4/1/2005 16:00	22.5	1.05	1.15	1.2	12	3108	0.300313	-0.54132	16.45201	0	0

Figure 5-Data

Here are the columns that are relevant to our analysis

- Column A: Underlying Symbol
- Column B: The price of the underlying option or X
- Column D: The option root
- Column E: Option Extension
- Column F: Option type (We are looking at Call options)
- Column G: The expiration date of the option, T
- Column H: The current date
- Column I: The strike price, K

- Column J: The last price this option was sold at. $C^{observed}$

After looking through the data and making sure all the formats were the same, an approximate 60 day period from 4/01/2004 to 6/30/2004 was chosen to focus on. In order to focus on relevant behavioral trends, simple macros in Visual Basic were used to filter out most of the irrelevant information and leave a small data set to work with. The first step was to design a macro to filter out all the unnecessary data and copy-paste my preferences all into one spreadsheet. Here is the code below

```
'Filter section code
'Filter options
    Windows(sName).Activate
    Range("T1").Select
    ActiveCell.FormulaR1C1 = "=ROUND(RC[-13]-RC[-12],0)"
    Range("T1").Select
    Selection.AutoFill Destination:=Range("T1:T155000")

    Range("U1").Select
    ActiveCell.FormulaR1C1 = "=ABS(RC[-12]/RC[-19]-1)"
    Range("U1").Select
    Selection.AutoFill Destination:=Range("U1:U155000")

    Range("A1").Select
    Selection.AutoFilter
    ActiveSheet.Range("$A$1:$U$155000").AutoFilter Field:=6, Criteria1:="call"
    ActiveSheet.Range("$A$1:$U$155000").AutoFilter Field:=7, Operator:= _
        xlFilterValues, Criteria2:=Array(1, "8/19/2005")
    ActiveSheet.Range("$A$1:$U$155000").AutoFilter Field:=21, Criteria1:=">=0" _
        , Operator:=xlAnd, Criteria2:="<=.03"
```

Figure 6- Filter Code

First two new columns were created; one to check the time to expiration (T-t), and one to calculate at-the-money options. It was decided to filter out options that were too close to the expiration date and too far in or out of the money. Then, each spreadsheet was filtered based on the following criteria:

1. Call options only
2. Each option was within 3% of the money
3. Time to maturity was greater than 3 weeks
4. Expiration dates were the same

With these criterion filtered, roughly 400 data points for each day was copied onto a separate file.

Next, an add-on was made to open each of the 60 files and apply the data-gathering process to each of them.

```
Sub Openfile()
Dim sPath As String, sName As String
Dim WB As Workbook


sPath = "C:\Users\Kshen\Desktop\thesis\Options data1\"
sName = Dir(sPath & "options_*.csv")

While sName <> ""
Set WB = Workbooks.Open(sPath & sName)

'Filter options go here

Windows(sName).Activate
ActiveWorkbook.Close (False)


sName = Dir()
Wend

End Sub
```

Figure 7-Data Gathering Code

This code was used to combine, opening and closing each file in the options data directory, and apply the filtering process then closing the files without altering the originals. However, the new table still contained over 25,000 rows. A pivot table was used to organize the Option root and Option extension rows to find out how many options had data-points throughout the 60-day period.

3	Row Labels	Count of Root
4	⊟JUL	**17**
5	HC	17
6	⊟NOV	**12**
7	HH	7
8	HI	5
9	⊟SUN	**41**
10	HA	10
11	HB	3
12	HS	9
13	HT	19
14	⊟A	**25**
15	HD	2
16	HE	13
17	HX	10
18	⊟AA	**3**
19	HY	3
20	⊟AAF	**7**
21	HQ	7
22	⊟AAI	**1**
23	HB	1

Figure 8-Table of Options

Here is a small portion of the pivot table. Through this filter, six different options that maintained 3%-within-the-money throughout most of the 60 day period were chosen and the implied volatilities per day were calculated.

Here is a macro that utilizes excel solver to calculate the Implied volatility of a call option. By utilizing the formula to calculate the $C^{observed}$ and matching that to the observed option price by changing the volatility

```
Sub Solve()
'
' Macro3 Macro
'
    Dim Row As Double
    Row = 2

    Do While Row < 66
    SolverReset
        SolverOk SetCell:="$W$" & Row, MaxMinVal:=2, ByChange:="$V$" & Row
        SolverAdd CellRef:="$W$" & Row, Relation:=2, FormulaText:="$J$" & Row

        SolverSave SaveArea:="$AF$5:$AF$8"
    SolverOptions MaxTime:=120, Iterations:=200, Precision:=0.00001, AssumeLinear:= _
        False, StepThru:=False, Estimates:=1, Derivatives:=1, SearchOption:=1, _
        IntTolerance:=100, Scaling:=False, Convergence:=0.0001, AssumeNonNeg:=True
        SolverSolve (True)


    Row = Row + 1
    Range("V" & Row).Select
    Loop

End Sub
```

Figure 9- Implied Volatility Solver Code

With the daily implied volatility calculated, the least squares method is applied to calculate the slope, a, and the intercept, b, in the LMMR.

$$a = \frac{(\sum x)(\sum y) - n(\sum x\, y)}{(\sum x)^2 - n\left(\sum x^2\right)}$$

$$b = \frac{(\sum x)(\sum x\, y) - (\sum y)(\sum x^2)}{(\sum x)^2 - n(\sum x^2)}$$

Where n is the total number of data points, y is the implied volatility, and x is the day number. Here are the scatter plots with the least square estimators,

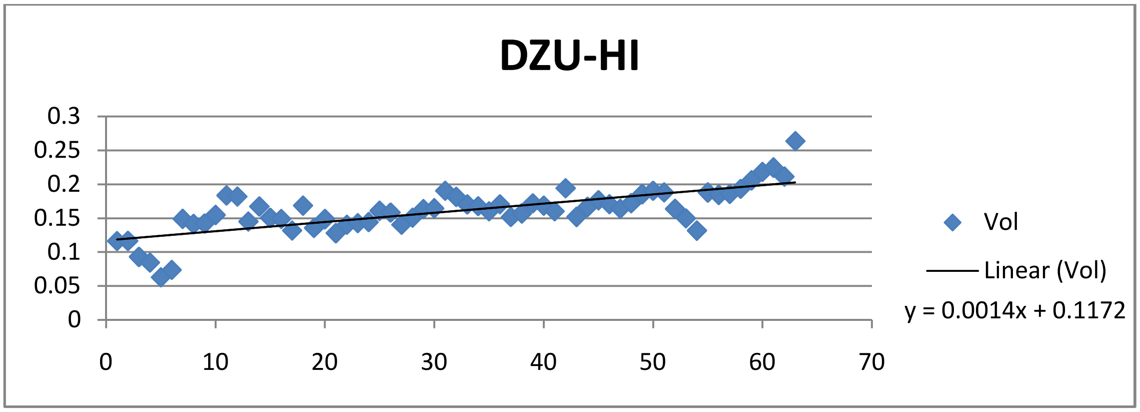

Figure 10-Graph of Option DZU-HI

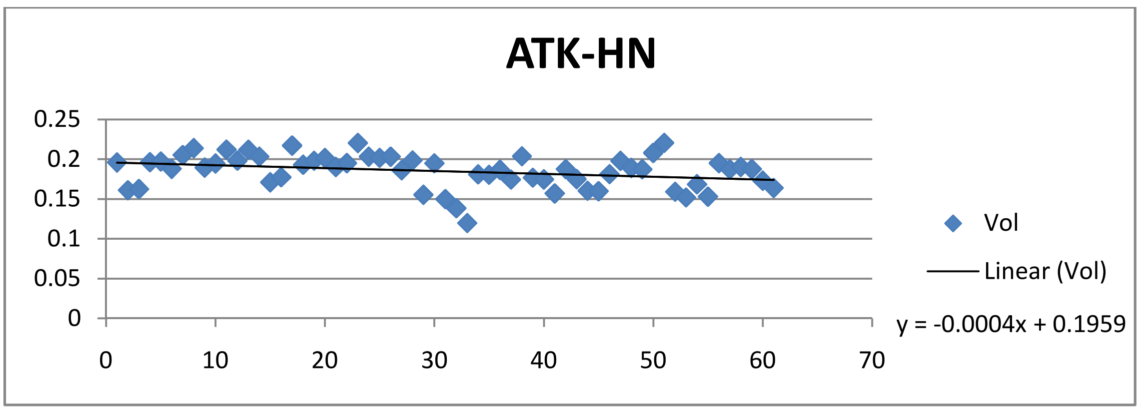

Figure 11-Graph of Option ATK-HN

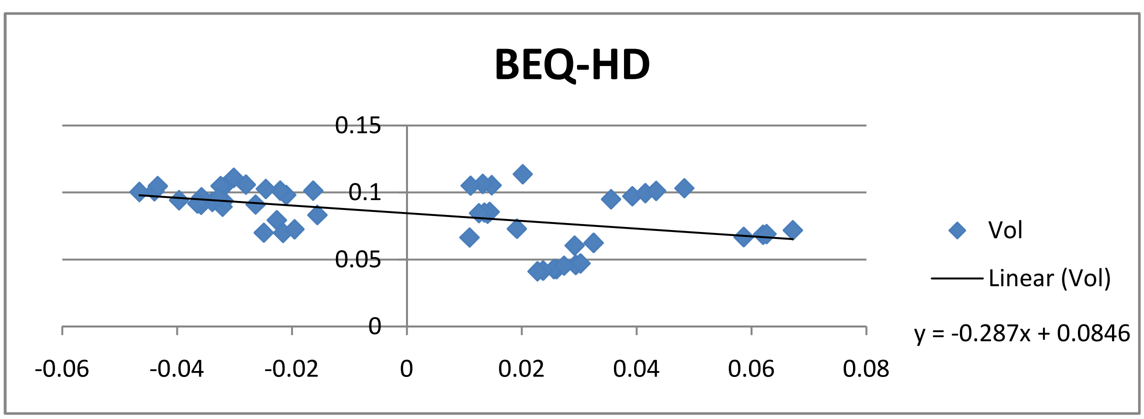

Figure 12-Graph of Option BEQ-HD

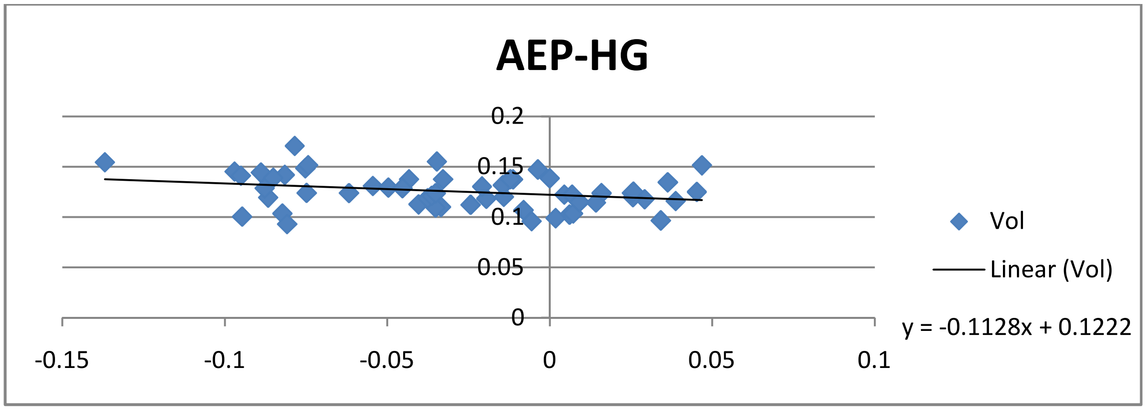

Figure 13-Graph of Option AEP-HG

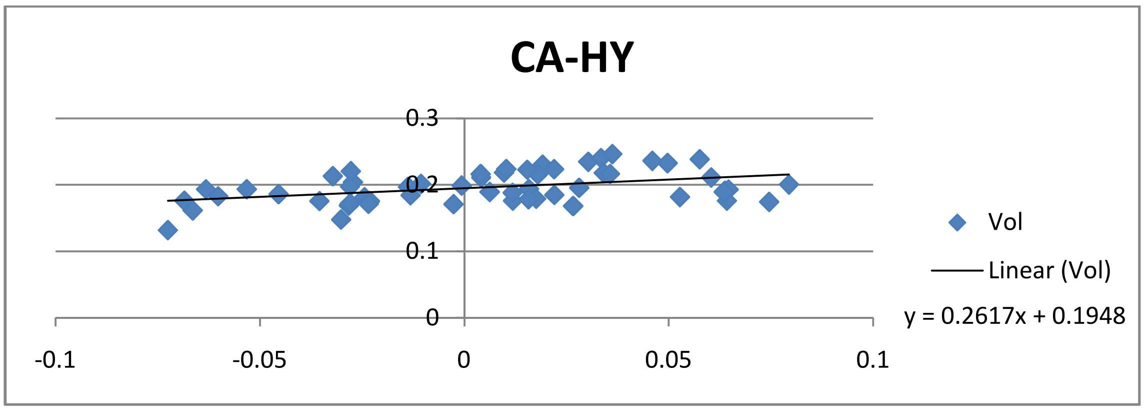

Figure 14-Graph of Option CA-HY

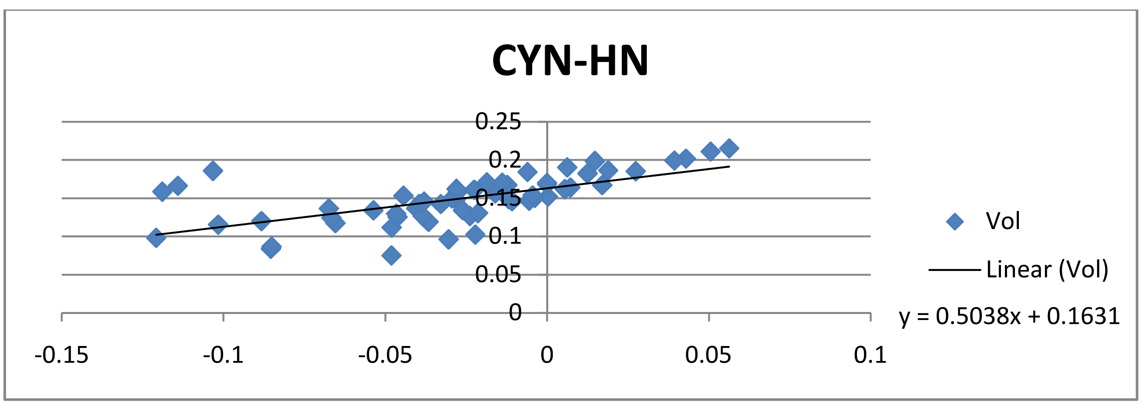

Figure 15-Graph of Option CYN-HN

Effective Volatility Calculation

We have that:

$$\bar{\sigma}^2 = \langle f^2 \rangle = \langle e^{2y} \rangle$$

This is an application of the centering condition on e^{2y}

$$\langle e^{2y} \rangle = \int_{-\infty}^{\infty} e^{2y} \Phi(y) \partial y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2y} \mathrm{e}^{-\mathrm{y}^2/2} \, \partial y$$

We will apply the change of variables $u = e^{2y}, \partial u = 2e^{2y}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2y} \mathrm{e}^{-\mathrm{y}^2/2} \, \partial y = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{y}^2/2} \, \partial e^{2y}$$

Apply Integration by parts: $u = \frac{1}{2\sqrt{2\pi}} \mathrm{e}^{-\mathrm{y}^2/2}, \partial v = \partial e^{2y}$

$$= \frac{1}{2\sqrt{2\pi}} \mathrm{e}^{-\mathrm{y}^2/2} e^{2y} \Big|_{-\infty}^{\infty} - \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2y} e^{-y^2/2} (-y) \partial y$$

The first equation is zero because $\mathrm{e}^{-\mathrm{y}^2/2}$ goes to 0 faster than e^{2y} approaches infinity.

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (y) e^{-y^2/2 + 2y} \partial y$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (y) e^{-1/2(y^2 - 4y)} \partial y$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (y) e^{-1/2(y-2)^2 + 2} \partial y$$

$$= \frac{e^2}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (y) e^{-1/2(y-2)^2} \partial y$$

Apply change of variables $u = y - 2, \partial u = 1$

$$= \frac{e^2}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (u + 2) \, e^{-u^2/2} \partial u$$

$$= \frac{e^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \partial u + \frac{e^2}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (u) \, e^{-u^2/2} \partial u$$

The first equation is equal to $e^2 \int \Phi(\mathrm{u}) \, \partial u = 1 * e^2$

$$= e^2 + \frac{e^2}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} (u)\, e^{-u^2/2} \partial u$$

Apply another change of variables $v = u^2, \partial v = 2u$

$$= e^2 + \frac{e^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \partial u^2$$

$$= e^2 + \frac{e^2}{\sqrt{2\pi}} e^{-u^2/2}(-2) \Big|_{-\infty}^{\infty}$$

$$= e^2$$

Thus we have that $\bar{\sigma}^2 = e^2$

Data

Option DZU-HI

Underlying symbol	X	T	t	K	Last	T-t	ITM	Vol.
ACDO	44.23	8/19/2005	4/1/2005 16:00	45	1.4	139	0.017409	0.116143
ACDO	44.27	8/19/2005	4/4/2005 16:00	45	1.4	136	0.01649	0.116284
ACDO	44.87	8/19/2005	4/5/2005 16:00	45	1.4	135	0.002897	0.09276
ACDO	45.07	8/19/2005	4/6/2005 16:00	45	1.4	134	0.001553	0.084347
ACDO	45.5	8/19/2005	4/7/2005 16:00	45	1.4	133	0.010989	0.062707
ACDO	45.31	8/19/2005	4/8/2005 16:00	45	1.4	132	0.006842	0.073599
ACDO	45.18	8/19/2005	4/11/2005 16:00	45	2.25	129	0.003984	0.148794
ACDO	45.36	8/19/2005	4/12/2005 16:00	45	2.25	128	0.007937	0.141233
ACDO	45.36	8/19/2005	4/13/2005 16:00	45	2.25	127	0.007937	0.141944
ACDO	45.1	8/19/2005	4/14/2005 16:00	45	2.25	126	0.002217	0.154628
ACDO	44.45	8/19/2005	4/15/2005 16:00	45	2.25	125	0.012373	0.183404
ACDO	44.55	8/19/2005	4/18/2005 16:00	45	2.25	122	0.010101	0.18184
ACDO	45.39	8/19/2005	4/19/2005 16:00	45	2.25	121	0.008592	0.144936
ACDO	44.93	8/19/2005	4/20/2005 16:00	45	2.25	120	0.001558	0.167051
ACDO	45.31	8/19/2005	4/21/2005 16:00	45	2.25	119	0.006842	0.150327
ACDO	45.36	8/19/2005	4/22/2005 16:00	45	2.25	118	0.007937	0.148712
ACDO	45.74	8/19/2005	4/25/2005 16:00	45	2.25	115	0.016178	0.131696
ACDO	44.55	8/19/2005	4/26/2005 16:00	45	2	114	0.010101	0.168378
ACDO	45.11	8/19/2005	4/27/2005 16:00	45	1.9	113	0.002438	0.135508
ACDO	44.85	8/19/2005	4/28/2005 16:00	45	1.9	112	0.003344	0.148395
ACDO	45.3	8/19/2005	4/29/2005 16:00	45	1.9	111	0.006623	0.127727
ACDO	45.27	8/19/2005	5/2/2005 16:00	45	2	108	0.005964	0.140127
ACDO	45.24	8/19/2005	5/3/2005 16:00	45	2	107	0.005305	0.142449
ACDO	45.48	8/19/2005	5/4/2005 16:00	45	2.15	106	0.010554	0.144058
ACDO	45.17	8/19/2005	5/5/2005 16:00	45	2.15	105	0.003764	0.160661
ACDO	45.24	8/19/2005	5/6/2005 16:00	45	2.15	104	0.005305	0.158112
ACDO	45.63	8/19/2005	5/9/2005 16:00	45	2.15	101	0.013807	0.140326
ACDO	45.46	8/19/2005	5/10/2005 16:00	45	2.15	100	0.010119	0.150465
ACDO	45.23	8/19/2005	5/11/2005 16:00	45	2.15	99	0.005085	0.163417
ACDO	45.23	8/19/2005	5/12/2005 16:00	45	2.15	98	0.005085	0.164418
ACDO	44.73	8/19/2005	5/13/2005 16:00	45	2.15	97	0.006036	0.19018
ACDO	44.98	8/19/2005	5/16/2005 16:00	45	2.15	94	0.000445	0.181352
ACDO	45.22	8/19/2005	5/17/2005 16:00	45	2.15	93	0.004865	0.170176
ACDO	45.29	8/19/2005	5/18/2005 16:00	45	2.15	92	0.006403	0.167567
ACDO	45.45	8/19/2005	5/19/2005 16:00	45	2.15	91	0.009901	0.15996
ACDO	45.28	8/19/2005	5/20/2005 16:00	45	2.15	90	0.006184	0.170314
ACDO	45.67	8/19/2005	5/23/2005 16:00	45	2.15	87	0.01467	0.151624

ACDO	45.6	8/19/2005	5/24/2005 16:00	45	2.15	86	0.013158	0.156849
ACDO	45.36	8/19/2005	5/25/2005 16:00	45	2.15	85	0.007937	0.171731
ACDO	45.44	8/19/2005	5/26/2005 16:00	45	2.15	84	0.009683	0.168406
ACDO	45.35	8/19/2005	5/27/2005 16:00	45	2	83	0.007718	0.160055
ACDO	44.82	8/19/2005	5/31/2005 16:00	45	2	79	0.004016	0.194106
ACDO	45.16	8/19/2005	6/1/2005 16:00	45	1.75	78	0.003543	0.151821
ACDO	44.92	8/19/2005	6/2/2005 16:00	45	1.75	77	0.001781	0.166324
ACDO	44.76	8/19/2005	6/3/2005 16:00	45	1.75	76	0.005362	0.176211
ACDO	44.94	8/19/2005	6/6/2005 16:00	45	1.75	73	0.001335	0.170449
ACDO	45.08	8/19/2005	6/7/2005 16:00	45	1.75	72	0.001775	0.163856
ACDO	44.96	8/19/2005	6/8/2005 16:00	45	1.75	71	0.00089	0.172078
ACDO	44.75	8/19/2005	6/9/2005 16:00	45	1.75	70	0.005587	0.185257
ACDO	43.96	8/19/2005	6/10/2005 16:00	45	1.4	69	0.023658	0.19026
ACDO	44.1	8/19/2005	6/13/2005 16:00	45	1.4	66	0.020408	0.187954
ACDO	44.28	8/19/2005	6/14/2005 16:00	45	1.25	65	0.01626	0.163524
ACDO	44.57	8/19/2005	6/15/2005 16:00	45	1.25	64	0.009648	0.149506
ACDO	44.91	8/19/2005	6/16/2005 16:00	45	1.25	63	0.002004	0.131492
ACDO	44.93	8/19/2005	6/17/2005 16:00	45	1.75	62	0.001558	0.187762
ACDO	45.07	8/19/2005	6/20/2005 16:00	45	1.75	59	0.001553	0.184399
ACDO	45.08	8/19/2005	6/21/2005 16:00	45	1.75	58	0.001775	0.18556
ACDO	44.99	8/19/2005	6/22/2005 16:00	45	1.75	57	0.000222	0.193107
ACDO	44.82	8/19/2005	6/23/2005 16:00	45	1.75	56	0.004016	0.205669
ACDO	44.65	8/19/2005	6/24/2005 16:00	45	1.75	55	0.007839	0.218181
ACDO	44.25	8/19/2005	6/27/2005 16:00	45	1.55	52	0.016949	0.224447
ACDO	44.51	8/19/2005	6/28/2005 16:00	45	1.55	51	0.011009	0.211192
ACDO	45.45	8/19/2005	6/29/2005 16:00	45	2.45	50	0.009901	0.263463

Option ATK-HN

Underlying symbol	X	T	t	K	Last	T-t	ITM	Vol.
ATK	70.44	8/19/2005	4/1/2005 16:00	70	4.7	139	0.006246	0.19582
ATK	71.69	8/19/2005	4/4/2005 16:00	70	4.7	136	0.023574	0.160922
ATK	71.67	8/19/2005	4/5/2005 16:00	70	4.7	135	0.023301	0.162327
ATK	71.9	8/19/2005	4/6/2005 16:00	70	5.5	134	0.026426	0.19616
ATK	71.9	8/19/2005	4/7/2005 16:00	70	5.5	133	0.026426	0.197057
ATK	71.9	8/19/2005	4/8/2005 16:00	70	5.3	132	0.026426	0.187808
ATK	71.44	8/19/2005	4/11/2005 16:00	70	5.3	129	0.020157	0.205088
ATK	71.19	8/19/2005	4/12/2005 16:00	70	5.3	128	0.016716	0.213749
ATK	70.85	8/19/2005	4/13/2005 16:00	70	4.6	127	0.011997	0.189374
ATK	70.01	8/19/2005	4/14/2005 16:00	70	4.2	126	0.000143	0.194477
ATK	69.97	8/19/2005	4/15/2005 16:00	70	4.5	125	0.000429	0.211878
ATK	70.37	8/19/2005	4/18/2005 16:00	70	4.4	122	0.005258	0.198071
ATK	69.936	8/19/2005	4/19/2005 16:00	70	4.4	121	0.000915	0.211703
ATK	69.54	8/19/2005	4/20/2005 16:00	70	4	120	0.006615	0.203105
ATK	70.67	8/19/2005	4/21/2005 16:00	70	4	119	0.009481	0.17088
ATK	70.49	8/19/2005	4/22/2005 16:00	70	4	118	0.006951	0.177297
ATK	70.15	8/19/2005	4/25/2005 16:00	70	4.5	115	0.002138	0.217022
ATK	70.1	8/19/2005	4/26/2005 16:00	70	4	114	0.001427	0.192858
ATK	69.96	8/19/2005	4/27/2005 16:00	70	4	113	0.000572	0.198035
ATK	69.51	8/19/2005	4/28/2005 16:00	70	3.8	112	0.007049	0.201439
ATK	69.18	8/19/2005	4/29/2005 16:00	70	3.4	111	0.011853	0.190153
ATK	69.69	8/19/2005	5/2/2005 16:00	70	3.7	108	0.004448	0.194921
ATK	68.85	8/19/2005	5/3/2005 16:00	70	3.7	107	0.016703	0.220267
ATK	69.86	8/19/2005	5/4/2005 16:00	70	3.9	106	0.002004	0.203035
ATK	68.18	8/19/2005	5/5/2005 16:00	70	3	105	0.026694	0.201612
ATK	69.18	8/19/2005	5/6/2005 16:00	70	3.5	104	0.011853	0.203085
ATK	68.9	8/19/2005	5/9/2005 16:00	70	3	101	0.015965	0.186054
ATK	68.5	8/19/2005	5/10/2005 16:00	70	3	100	0.021898	0.198421
ATK	70	8/19/2005	5/11/2005 16:00	70	3	99	0	0.155167
ATK	69.11	8/19/2005	5/12/2005 16:00	70	3.2	98	0.012878	0.194798
ATK	69.19	8/19/2005	5/16/2005 16:00	70	2.4	94	0.011707	0.149825
ATK	69.6	8/19/2005	5/17/2005 16:00	70	2.4	93	0.005747	0.138338
ATK	70.2	8/19/2005	5/18/2005 16:00	70	2.4	92	0.002849	0.119733
ATK	69.64	8/19/2005	5/19/2005 16:00	70	3.1	91	0.005169	0.180785
ATK	69.69	8/19/2005	5/20/2005 16:00	70	3.1	90	0.004448	0.180353
ATK	69.8	8/19/2005	5/23/2005 16:00	70	3.2	87	0.002865	0.186458
ATK	70.2	8/19/2005	5/24/2005 16:00	70	3.2	86	0.002849	0.174253
ATK	69.35	8/19/2005	5/25/2005 16:00	70	3.2	85	0.009373	0.203619
ATK	70.2	8/19/2005	5/26/2005 16:00	70	3.2	84	0.002849	0.176673

ATK	71.15	8/19/2005	5/27/2005 16:00	70	3.7	83	0.016163	0.174497
ATK	71.7	8/19/2005	5/31/2005 16:00	70	3.7	79	0.02371	0.156892
ATK	71.48	8/19/2005	6/1/2005 16:00	70	4	78	0.020705	0.187412
ATK	71.01	8/19/2005	6/2/2005 16:00	70	3.5	77	0.014223	0.17483
ATK	71.4	8/19/2005	6/3/2005 16:00	70	3.5	76	0.019608	0.160135
ATK	71.5	8/19/2005	6/6/2005 16:00	70	3.5	73	0.020979	0.159704
ATK	71.03	8/19/2005	6/7/2005 16:00	70	3.5	72	0.014501	0.181047
ATK	70.65	8/19/2005	6/8/2005 16:00	70	3.5	71	0.0092	0.197775
ATK	70.91	8/19/2005	6/9/2005 16:00	70	3.5	70	0.012833	0.188976
ATK	70.99	8/19/2005	6/10/2005 16:00	70	3.5	69	0.013946	0.187257
ATK	70.61	8/19/2005	6/13/2005 16:00	70	3.5	66	0.008639	0.207791
ATK	70.34	8/19/2005	6/14/2005 16:00	70	3.5	65	0.004834	0.220382
ATK	70.32	8/19/2005	6/15/2005 16:00	70	2.6	64	0.004551	0.158974
ATK	70.53	8/19/2005	6/16/2005 16:00	70	2.6	63	0.007515	0.151789
ATK	70.34	8/19/2005	6/17/2005 16:00	70	2.7	62	0.004834	0.168379
ATK	70.81	8/19/2005	6/20/2005 16:00	70	2.7	59	0.011439	0.152887
ATK	70.4	8/19/2005	6/21/2005 16:00	70	3	58	0.005682	0.194905
ATK	70.27	8/19/2005	6/22/2005 16:00	70	2.8	57	0.003842	0.187217
ATK	69.39	8/19/2005	6/23/2005 16:00	70	2.35	56	0.008791	0.190146
ATK	68.5	8/19/2005	6/27/2005 16:00	70	1.8	52	0.021898	0.1874
ATK	69.17	8/19/2005	6/28/2005 16:00	70	1.9	51	0.011999	0.172795
ATK	69.65	8/19/2005	6/29/2005 16:00	70	2	50	0.005025	0.163724

Option BEQ-HD

Underlying symbol	X	T	t	K	Last	T-t	ITM	Vol
BKST	19.86	8/19/2005	4/15/2005 16:00	20	0.5	125	0.007049	0.084014
BKST	19.86	8/19/2005	4/18/2005 16:00	20	0.5	122	0.007049	0.085477
BKST	19.87	8/19/2005	4/19/2005 16:00	20	0.5	121	0.006543	0.085009
BKST	19.88	8/19/2005	4/20/2005 16:00	20	0.5	120	0.006036	0.084534
BKST	19.86	8/19/2005	4/21/2005 16:00	20	0.6	119	0.007049	0.105329
BKST	19.82	8/19/2005	4/22/2005 16:00	20	0.4	118	0.009082	0.072925
BKST	19.9	8/19/2005	4/25/2005 16:00	20	0.4	115	0.005025	0.066424
BKST	19.88	8/19/2005	4/26/2005 16:00	20	0.6	114	0.006036	0.106372
BKST	19.9	8/19/2005	4/27/2005 16:00	20	0.6	113	0.005025	0.104983
BKST	19.82	8/19/2005	4/28/2005 16:00	20	0.6	112	0.009082	0.113544
BKST	19.77	8/19/2005	4/29/2005 16:00	20	0.2	111	0.011634	0.04287
BKST	19.78	8/19/2005	5/2/2005 16:00	20	0.2	108	0.011122	0.042975
BKST	19.75	8/19/2005	5/3/2005 16:00	20	0.2	107	0.012658	0.046013
BKST	19.8	8/19/2005	5/4/2005 16:00	20	0.2	106	0.010101	0.041808
BKST	19.81	8/19/2005	5/5/2005 16:00	20	0.2	105	0.009591	0.041205
BKST	19.75	8/19/2005	5/6/2005 16:00	20	0.2	104	0.012658	0.047087
BKST	19.78	8/19/2005	5/9/2005 16:00	20	0.2	101	0.011122	0.045453
BKST	19.51	8/19/2005	5/10/2005 16:00	20	0.2	100	0.025115	0.068506
BKST	19.51	8/19/2005	5/11/2005 16:00	20	0.2	99	0.025115	0.068969
BKST	19.48	8/19/2005	5/12/2005 16:00	20	0.2	98	0.026694	0.071771
BKST	19.55	8/19/2005	5/13/2005 16:00	20	0.2	97	0.023018	0.066743
BKST	19.64	8/19/2005	5/16/2005 16:00	20	0.4	94	0.01833	0.103129
BKST	19.71	8/19/2005	5/17/2005 16:00	20	0.4	93	0.014713	0.097136
BKST	19.74	8/19/2005	5/18/2005 16:00	20	0.4	92	0.013171	0.094863
BKST	19.7	8/19/2005	5/19/2005 16:00	20	0.4	91	0.015228	0.099471
BKST	19.69	8/19/2005	5/20/2005 16:00	20	0.4	90	0.015744	0.101147
BKST	19.775	8/19/2005	5/23/2005 16:00	20	0.25	87	0.011378	0.062419
BKST	19.8	8/19/2005	5/24/2005 16:00	20	0.25	86	0.010101	0.060412
BKST	20.18	8/19/2005	5/25/2005 16:00	20	0.6	85	0.00892	0.090975
BKST	20.11	8/19/2005	5/26/2005 16:00	20	0.6	84	0.00547	0.101253
BKST	20.14	8/19/2005	5/27/2005 16:00	20	0.6	83	0.006951	0.098049
BKST	20.14	8/19/2005	5/31/2005 16:00	20	0.6	79	0.006951	0.101351
BKST	20.2	8/19/2005	6/1/2005 16:00	20	0.6	78	0.009901	0.093607
BKST	20.21	8/19/2005	6/2/2005 16:00	20	0.6	77	0.010391	0.092942
BKST	20.15	8/19/2005	6/3/2005 16:00	20	0.6	76	0.007444	0.102568
BKST	20.21	8/19/2005	6/6/2005 16:00	20	0.6	73	0.010391	0.096407
BKST	20.23	8/19/2005	6/7/2005 16:00	20	0.6	72	0.011369	0.094182
BKST	20.16	8/19/2005	6/8/2005 16:00	20	0.6	71	0.007937	0.105776
BKST	20.18	8/19/2005	6/9/2005 16:00	20	0.6	70	0.00892	0.103773

BKST	20.18	8/19/2005	6/10/2005 16:00	20	0.6	69	0.00892	0.104755
BKST	20.16	8/19/2005	6/13/2005 16:00	20	0.6	66	0.007937	0.110872
BKST	20.23	8/19/2005	6/14/2005 16:00	20	0.6	65	0.011369	0.100913
BKST	20.24	8/19/2005	6/15/2005 16:00	20	0.6	64	0.011858	0.10029
BKST	20.22	8/19/2005	6/16/2005 16:00	20	0.6	63	0.01088	0.104668
BKST	20.16	8/19/2005	6/17/2005 16:00	20	0.5	62	0.007937	0.089372
BKST	20.17	8/19/2005	6/20/2005 16:00	20	0.5	59	0.008428	0.090717
BKST	20.17	8/19/2005	6/21/2005 16:00	20	0.5	58	0.008428	0.091759
BKST	20.15	8/19/2005	6/22/2005 16:00	20	0.5	57	0.007444	0.096192
BKST	20.07	8/19/2005	6/23/2005 16:00	20	0.4	56	0.003488	0.083165
BKST	20.1	8/19/2005	6/24/2005 16:00	20	0.4	55	0.004975	0.079258
BKST	20.09	8/19/2005	6/27/2005 16:00	20	0.35	52	0.00448	0.069839
BKST	20.08	8/19/2005	6/28/2005 16:00	20	0.35	51	0.003984	0.072539
BKST	20.1	8/19/2005	6/29/2005 16:00	20	0.35	50	0.004975	0.069988

Option AEP-HG

Underlying symbol	X	T	t	K	Last	T-t	ITM	Vol
AEP	34.13	8/19/2005	4/1/2005 16:00	35	1.05	139	0.025491	0.124939
AEP	34.36	8/19/2005	4/5/2005 16:00	35	0.85	135	0.018626	0.096576
AEP	34.28	8/19/2005	4/6/2005 16:00	35	1	134	0.021004	0.115827
AEP	34.46	8/19/2005	4/7/2005 16:00	35	1.1	133	0.01567	0.117711
AEP	34.53	8/19/2005	4/8/2005 16:00	35	1.15	132	0.013611	0.119791
AEP	34.89	8/19/2005	4/11/2005 16:00	35	1.15	129	0.003153	0.102453
AEP	35.1	8/19/2005	4/12/2005 16:00	35	1.2	128	0.002849	0.095895
AEP	34.92	8/19/2005	4/13/2005 16:00	35	1.35	127	0.002291	0.122196
AEP	34.72	8/19/2005	4/14/2005 16:00	35	1.25	126	0.008065	0.123638
AEP	34.19	8/19/2005	4/15/2005 16:00	35	1.25	125	0.023691	0.151398
AEP	34.57	8/19/2005	4/18/2005 16:00	35	1.15	122	0.012439	0.123861
AEP	34.76	8/19/2005	4/19/2005 16:00	35	1.15	121	0.006904	0.114286
AEP	34.57	8/19/2005	4/20/2005 16:00	35	1.15	120	0.012439	0.125162
AEP	34.88	8/19/2005	4/21/2005 16:00	35	1.1	119	0.00344	0.103519
AEP	34.97	8/19/2005	4/22/2005 16:00	35	1.1	118	0.000858	0.098833
AEP	35.13	8/19/2005	4/25/2005 16:00	35	1.25	115	0.003701	0.106836
AEP	34.89	8/19/2005	4/26/2005 16:00	35	1.25	114	0.003153	0.122016
AEP	35.31	8/19/2005	4/27/2005 16:00	35	1.45	113	0.008779	0.118245
AEP	35	8/19/2005	4/28/2005 16:00	35	1.45	112	0	0.138471
AEP	35.22	8/19/2005	4/29/2005 16:00	35	1.4	111	0.006246	0.120066
AEP	35.37	8/19/2005	5/2/2005 16:00	35	1.4	108	0.010461	0.112134
AEP	35.61	8/19/2005	5/3/2005 16:00	35	1.55	107	0.01713	0.112606
AEP	35.5	8/19/2005	5/4/2005 16:00	35	1.45	106	0.014085	0.109957
AEP	35.52	8/19/2005	5/5/2005 16:00	35	1.45	105	0.01464	0.109192
AEP	35.21	8/19/2005	5/6/2005 16:00	35	1.45	104	0.005964	0.131577
AEP	35.16	8/19/2005	5/9/2005 16:00	35	1.45	101	0.004551	0.137346
AEP	35.17	8/19/2005	5/10/2005 16:00	35	1.45	100	0.004834	0.137544
AEP	35.29	8/19/2005	5/11/2005 16:00	35	1.45	99	0.008218	0.130295
AEP	35.05	8/19/2005	5/12/2005 16:00	35	1.45	98	0.001427	0.14715
AEP	34.51	8/19/2005	5/13/2005 16:00	35	1.05	97	0.014199	0.13446
AEP	34.88	8/19/2005	5/16/2005 16:00	35	1.05	94	0.00344	0.114789
AEP	35.43	8/19/2005	5/17/2005 16:00	35	1.55	93	0.012137	0.137468
AEP	35.45	8/19/2005	5/18/2005 16:00	35	1.7	92	0.012694	0.155071
AEP	35.7	8/19/2005	5/19/2005 16:00	35	1.65	91	0.019608	0.130778
AEP	35.55	8/19/2005	5/20/2005 16:00	35	1.6	90	0.015471	0.137387
AEP	35.61	8/19/2005	5/23/2005 16:00	35	1.55	87	0.01713	0.129261
AEP	35.55	8/19/2005	5/24/2005 16:00	35	1.5	86	0.015471	0.128709
AEP	35.45	8/19/2005	5/25/2005 16:00	35	1.35	85	0.012694	0.11854
AEP	35.43	8/19/2005	5/26/2005 16:00	35	1.35	84	0.012137	0.121049

AEP	35.41	8/19/2005	5/27/2005 16:00	35	1.35	83	0.011579	0.123573
AEP	35.69	8/19/2005	5/31/2005 16:00	35	1.5	79	0.019333	0.123809
AEP	35.91	8/19/2005	6/1/2005 16:00	35	1.5	78	0.025341	0.1035
AEP	35.82	8/19/2005	6/2/2005 16:00	35	1.75	77	0.022892	0.148439
AEP	35.8	8/19/2005	6/3/2005 16:00	35	1.75	76	0.022346	0.151437
AEP	35.92	8/19/2005	6/6/2005 16:00	35	1.75	73	0.025612	0.144072
AEP	35.8	8/19/2005	6/7/2005 16:00	35	1.85	72	0.022346	0.170509
AEP	35.82	8/19/2005	6/8/2005 16:00	35	1.65	71	0.022892	0.141812
AEP	35.94	8/19/2005	6/9/2005 16:00	35	1.45	70	0.026155	0.100379
AEP	35.73	8/19/2005	6/10/2005 16:00	35	1.45	69	0.020431	0.123903
AEP	35.81	8/19/2005	6/13/2005 16:00	35	1.45	66	0.022619	0.11929
AEP	35.88	8/19/2005	6/15/2005 16:00	35	1.65	64	0.024526	0.145186
AEP	35.72	8/19/2005	6/16/2005 16:00	35	1.2	63	0.020157	0.093042
AEP	35.77	8/19/2005	6/17/2005 16:00	35	1.45	62	0.021526	0.128504
AEP	35.71	8/19/2005	6/20/2005 16:00	35	1.45	59	0.019882	0.138878
AEP	35.78	8/19/2005	6/21/2005 16:00	35	1.5	58	0.0218	0.140947
AEP	36.07	8/19/2005	6/24/2005 16:00	35	1.75	55	0.029665	0.154377

Option CA-HY

Underlying symbol	X	T	t	K	Last	T-t	ITM	Vol
CA	27.26	8/19/2005	4/1/2005 16:00	27.5	1.6	139	0.008804	0.193726
CA	27.54	8/19/2005	4/4/2005 16:00	27.5	1.55	136	0.001452	0.170794
CA	27.51	8/19/2005	4/5/2005 16:00	27.5	1.75	135	0.000364	0.19876
CA	27.44	8/19/2005	4/6/2005 16:00	27.5	1.8	134	0.002187	0.210827
CA	27.86	8/19/2005	4/7/2005 16:00	27.5	1.8	133	0.012922	0.18128
CA	27.36	8/19/2005	4/8/2005 16:00	27.5	1.8	132	0.005117	0.218258
CA	27.23	8/19/2005	4/11/2005 16:00	27.5	1.8	129	0.009916	0.230192
CA	27.65	8/19/2005	4/12/2005 16:00	27.5	1.8	128	0.005425	0.201362
CA	28	8/19/2005	4/13/2005 16:00	27.5	1.8	127	0.017857	0.17541
CA	27.95	8/19/2005	4/14/2005 16:00	27.5	2.05	126	0.0161	0.212922
CA	27.2	8/19/2005	4/15/2005 16:00	27.5	1.7	125	0.011029	0.223395
CA	27.26	8/19/2005	4/18/2005 16:00	27.5	1.65	122	0.008804	0.215712
CA	27.1	8/19/2005	4/19/2005 16:00	27.5	1.7	121	0.01476	0.234567
CA	26.9	8/19/2005	4/20/2005 16:00	27.5	1.6	120	0.022305	0.235976
CA	27.3	8/19/2005	4/21/2005 16:00	27.5	1.7	119	0.007326	0.222654
CA	27.07	8/19/2005	4/22/2005 16:00	27.5	1.7	118	0.015885	0.24003
CA	27.45	8/19/2005	4/25/2005 16:00	27.5	1.7	115	0.001821	0.216012
CA	27.05	8/19/2005	4/26/2005 16:00	27.5	1.7	114	0.016636	0.246162
CA	27.08	8/19/2005	4/27/2005 16:00	27.5	1.5	113	0.01551	0.217693
CA	26.8	8/19/2005	4/28/2005 16:00	27.5	1.5	112	0.026119	0.238279
CA	26.9	8/19/2005	4/29/2005 16:00	27.5	1.5	111	0.022305	0.232569
CA	27.08	8/19/2005	5/2/2005 16:00	27.5	1.45	108	0.01551	0.216327
CA	27.38	8/19/2005	5/3/2005 16:00	27.5	1.65	107	0.004383	0.223467
CA	27.82	8/19/2005	5/4/2005 16:00	27.5	1.75	106	0.011503	0.203824
CA	27.66	8/19/2005	5/5/2005 16:00	27.5	1.6	105	0.005785	0.196777
CA	27.82	8/19/2005	5/6/2005 16:00	27.5	1.85	104	0.011503	0.220399
CA	28.01	8/19/2005	5/9/2005 16:00	27.5	1.7	101	0.018208	0.185828
CA	27.81	8/19/2005	5/10/2005 16:00	27.5	1.65	100	0.011147	0.197139
CA	27.81	8/19/2005	5/11/2005 16:00	27.5	1.45	99	0.011147	0.168931
CA	27.75	8/19/2005	5/12/2005 16:00	27.5	1.45	98	0.009009	0.175171
CA	27.8	8/19/2005	5/13/2005 16:00	27.5	1.45	97	0.010791	0.171898
CA	28.13	8/19/2005	5/16/2005 16:00	27.5	1.7	94	0.022396	0.182627
CA	28.21	8/19/2005	5/17/2005 16:00	27.5	1.7	93	0.025168	0.175958
CA	27.3	8/19/2005	5/27/2005 16:00	27.5	1.15	83	0.007326	0.184712
CA	27.27	8/19/2005	5/31/2005 16:00	27.5	1	79	0.008434	0.168003
CA	27.35	8/19/2005	6/1/2005 16:00	27.5	1.1	78	0.005484	0.178898
CA	27.4	8/19/2005	6/2/2005 16:00	27.5	1.1	77	0.00365	0.17588
CA	27	8/19/2005	6/3/2005 16:00	27.5	1.1	76	0.018519	0.210918
CA	26.87	8/19/2005	6/6/2005 16:00	27.5	0.95	73	0.023446	0.200327
CA	27	8/19/2005	6/7/2005 16:00	27.5	0.95	72	0.018519	0.19136

CA	27	8/19/2005	6/8/2005 16:00	27.5	0.95	71	0.018519	0.192867
CA	27.01	8/19/2005	6/9/2005 16:00	27.5	0.85	70	0.018141	0.175985
CA	26.94	8/19/2005	6/10/2005 16:00	27.5	0.8	69	0.020787	0.174245
CA	27.12	8/19/2005	6/13/2005 16:00	27.5	0.9	66	0.014012	0.18155
CA	27.05	8/19/2005	6/14/2005 16:00	27.5	0.9	65	0.016636	0.189187
CA	27.39	8/19/2005	6/15/2005 16:00	27.5	1	64	0.004016	0.178268
CA	27.71	8/19/2005	6/16/2005 16:00	27.5	1	63	0.007578	0.147654
CA	27.42	8/19/2005	6/17/2005 16:00	27.5	1.05	62	0.002918	0.187816
CA	27.46	8/19/2005	6/20/2005 16:00	27.5	1.05	59	0.001457	0.189146
CA	27.65	8/19/2005	6/21/2005 16:00	27.5	1.05	58	0.005425	0.171164
CA	27.92	8/19/2005	6/22/2005 16:00	27.5	1.15	57	0.015043	0.161395
CA	27.83	8/19/2005	6/23/2005 16:00	27.5	1.25	56	0.011858	0.193354
CA	27.58	8/19/2005	6/24/2005 16:00	27.5	1.05	55	0.002901	0.184081
CA	27.34	8/19/2005	6/27/2005 16:00	27.5	0.95	52	0.005852	0.195452
CA	27.91	8/19/2005	6/28/2005 16:00	27.5	0.95	51	0.01469	0.131747
CA	27.85	8/19/2005	6/29/2005 16:00	27.5	1.2	50	0.012567	0.193358

Option CYN-HN

Underlying symbol	X	T	t	K	Last	T-t	ITM	Vol
CYN	69.34	8/19/2005	4/1/2005 16:00	70	3.5	139	0.009518	0.167148
CYN	69.99	8/19/2005	4/4/2005 16:00	70	3.5	136	0.000143	0.152002
CYN	70.21	8/19/2005	4/5/2005 16:00	70	3.5	135	0.002991	0.146561
CYN	70.9	8/19/2005	4/6/2005 16:00	70	3.5	134	0.012694	0.126793
CYN	70.8	8/19/2005	4/7/2005 16:00	70	3.5	133	0.011299	0.130534
CYN	70.14	8/19/2005	4/8/2005 16:00	70	3.5	132	0.001996	0.150645
CYN	69.74	8/19/2005	4/11/2005 16:00	70	3.5	129	0.003728	0.164029
CYN	70.39	8/19/2005	4/12/2005 16:00	70	3.5	128	0.005541	0.146301
CYN	70.16	8/19/2005	4/13/2005 16:00	70	3.5	127	0.002281	0.153743
CYN	69.04	8/19/2005	4/14/2005 16:00	70	3.5	126	0.013905	0.185304
CYN	68.55	8/19/2005	4/18/2005 16:00	70	3.5	122	0.021152	0.201761
CYN	68.68	8/19/2005	4/19/2005 16:00	70	3.5	121	0.01922	0.199319
CYN	68.15	8/19/2005	4/21/2005 16:00	70	3.5	119	0.027146	0.215088
CYN	68.35	8/19/2005	4/22/2005 16:00	70	3.5	118	0.02414	0.210923
CYN	70	8/19/2005	4/25/2005 16:00	70	3.5	115	0.00	0.168186
CYN	69.4	8/19/2005	4/26/2005 16:00	70	3.5	114	0.008646	0.186424
CYN	70	8/19/2005	4/27/2005 16:00	70	3.5	113	0.000	0.169971
CYN	69.61	8/19/2005	4/28/2005 16:00	70	3.5	112	0.005603	0.182344
CYN	70.5	8/19/2005	4/29/2005 16:00	70	3.5	111	0.007092	0.156396
CYN	71	8/19/2005	5/2/2005 16:00	70	3.5	108	0.014085	0.142469
CYN	70.85	8/19/2005	5/3/2005 16:00	70	3.5	107	0.011997	0.148434
CYN	71.21	8/19/2005	5/4/2005 16:00	70	3.5	106	0.016992	0.136754
CYN	70.87	8/19/2005	5/5/2005 16:00	70	3.5	105	0.012276	0.149514
CYN	70.36	8/19/2005	5/6/2005 16:00	70	3.5	104	0.005117	0.167274
CYN	70.64	8/19/2005	5/9/2005 16:00	70	3.5	101	0.00906	0.161007
CYN	70.39	8/19/2005	5/10/2005 16:00	70	3.5	100	0.005541	0.170278
CYN	70.77	8/19/2005	5/11/2005 16:00	70	3.5	99	0.01088	0.158539
CYN	69.83	8/19/2005	5/12/2005 16:00	70	3.5	98	0.002434	0.190298
CYN	69.6	8/19/2005	5/13/2005 16:00	70	3.5	97	0.005747	0.198586
CYN	70.16	8/19/2005	5/16/2005 16:00	70	3.5	94	0.002281	0.184296
CYN	70.58	8/19/2005	5/17/2005 16:00	70	2.35	93	0.008218	0.102473
CYN	71.25	8/19/2005	5/18/2005 16:00	70	2.35	92	0.017544	0.075157
CYN	70.78	8/19/2005	5/19/2005 16:00	70	2.35	91	0.01102	0.096314
CYN	70.93	8/19/2005	5/20/2005 16:00	70	2.8	90	0.013112	0.119205
CYN	71.18	8/19/2005	5/23/2005 16:00	70	2.8	87	0.016578	0.111718
CYN	70.63	8/19/2005	5/24/2005 16:00	70	2.8	86	0.00892	0.13425
CYN	69.87	8/19/2005	5/25/2005 16:00	70	2.8	85	0.001861	0.161917
CYN	70.28	8/19/2005	5/26/2005 16:00	70	2.8	84	0.003984	0.148949
CYN	70.9	8/19/2005	5/27/2005 16:00	70	2.8	83	0.012694	0.126823

CYN	71.03	8/19/2005	5/31/2005 16:00	70	2.8	79	0.014501	0.125509
CYN	71.89	8/19/2005	6/1/2005 16:00	70	2.8	78	0.02629	0.084097
CYN	71.86	8/19/2005	6/2/2005 16:00	70	2.8	77	0.025884	0.086823
CYN	71	8/19/2005	6/3/2005 16:00	70	2.8	76	0.014085	0.129909
CYN	71.35	8/19/2005	6/6/2005 16:00	70	2.8	73	0.018921	0.117538
CYN	70.8	8/19/2005	6/7/2005 16:00	70	2.8	72	0.011299	0.142801
CYN	70.76	8/19/2005	6/8/2005 16:00	70	2.8	71	0.010741	0.145673
CYN	71.06	8/19/2005	6/9/2005 16:00	70	2.8	70	0.014917	0.134128
CYN	71.3	8/19/2005	6/10/2005 16:00	70	2.8	69	0.018233	0.124329
CYN	70.52	8/19/2005	6/13/2005 16:00	70	2.8	66	0.007374	0.162172
CYN	70.34	8/19/2005	6/14/2005 16:00	70	2.8	65	0.004834	0.170898
CYN	70.8	8/19/2005	6/15/2005 16:00	70	2.8	64	0.011299	0.153304
CYN	71.2	8/19/2005	6/16/2005 16:00	70	2.8	63	0.016854	0.136546
CYN	71.55	8/19/2005	6/17/2005 16:00	70	2.8	62	0.021663	0.120138
CYN	71.7	8/19/2005	6/20/2005 16:00	70	2.8	59	0.02371	0.115537
CYN	71.99	8/19/2005	6/21/2005 16:00	70	2.8	58	0.027643	0.098186
CYN	71.89	8/19/2005	6/23/2005 16:00	70	3.4	56	0.02629	0.158578
CYN	71.78	8/19/2005	6/24/2005 16:00	70	3.4	55	0.024798	0.166313
CYN	71.52	8/19/2005	6/27/2005 16:00	70	3.4	52	0.021253	0.185791

Modified Price & $\tilde{P}_1$

DZU-HI		ATK-HN		BEQ-HD		AEP-HG		CA-HY		CYN-HN	
$\tilde{P}_1$	$P_0-\tilde{P}_1$	$\tilde{P}_1$	$P_0-\tilde{P}_1$	$\tilde{P}_1$	$P_0-\tilde{P}_1$	$\tilde{P}_1$	$P_0-\tilde{P}_1$	$\tilde{P}_1$	$P_0-\tilde{P}_1$	$\tilde{P}_1$	$P_0-\tilde{P}_1$
-0.197	1.597	-1.972	6.672	-0.023	0.523	-0.266	1.316	-0.076	1.676	-2.286	5.786
-0.195	1.595	-1.908	6.608	-0.023	0.523	-0.264	1.114	-0.076	1.626	-2.277	5.777
-0.196	1.596	-1.903	6.603	-0.023	0.523	-0.263	1.263	-0.076	1.826	-2.270	5.770
-0.194	1.594	-1.911	7.411	-0.023	0.523	-0.263	1.363	-0.075	1.875	-2.241	5.741
-0.179	1.579	-1.904	7.404	-0.023	0.623	-0.263	1.413	-0.075	1.875	-2.239	5.739
-0.187	1.587	-1.892	7.192	-0.023	0.423	-0.260	1.410	-0.075	1.875	-2.245	5.745
-0.192	2.442	-1.890	7.190	-0.022	0.422	-0.256	1.456	-0.074	1.874	-2.214	5.714
-0.191	2.441	-1.890	7.190	-0.022	0.622	-0.258	1.608	-0.074	1.874	-2.211	5.711
-0.190	2.440	-1.882	6.482	-0.022	0.622	-0.257	1.507	-0.073	1.873	-2.204	5.704
-0.190	2.440	-1.880	6.080	-0.022	0.622	-0.253	1.503	-0.073	2.123	-2.166	5.666
-0.188	2.438	-1.872	6.372	-0.022	0.222	-0.253	1.403	-0.072	1.772	-2.109	5.609
-0.186	2.436	-1.850	6.250	-0.022	0.222	-0.252	1.402	-0.072	1.722	-2.107	5.607
-0.185	2.435	-1.842	6.242	-0.022	0.222	-0.251	1.401	-0.071	1.771	-2.063	5.563
-0.185	2.435	-1.831	5.831	-0.022	0.222	-0.250	1.350	-0.070	1.670	-2.064	5.564
-0.184	2.434	-1.822	5.822	-0.021	0.221	-0.248	1.348	-0.071	1.771	-2.098	5.598
-0.183	2.433	-1.818	5.818	-0.021	0.221	-0.243	1.493	-0.070	1.770	-2.073	5.573
-0.178	2.428	-1.798	6.298	-0.021	0.221	-0.245	1.495	-0.070	1.770	-2.080	5.580
-0.180	2.180	-1.790	5.790	-0.019	0.219	-0.239	1.689	-0.069	1.769	-2.062	5.562
-0.180	2.080	-1.782	5.782	-0.019	0.219	-0.242	1.692	-0.068	1.568	-2.064	5.564
-0.179	2.079	-1.769	5.569	-0.019	0.219	-0.239	1.639	-0.067	1.567	-2.023	5.523
-0.178	2.078	-1.756	5.156	-0.019	0.219	-0.232	1.632	-0.067	1.567	-2.021	5.521
-0.176	2.176	-1.740	5.440	-0.020	0.420	-0.225	1.775	-0.067	1.517	-1.992	5.492
-0.175	2.175	-1.716	5.416	-0.020	0.420	-0.226	1.676	-0.067	1.717	-2.003	5.503
-0.173	2.323	-1.726	5.626	-0.020	0.420	-0.225	1.675	-0.067	1.817	-2.001	5.501
-0.174	2.324	-1.675	4.675	-0.020	0.420	-0.232	1.682	-0.067	1.667	-1.972	5.472
-0.173	2.323	-1.699	5.199	-0.020	0.420	-0.229	1.679	-0.067	1.917	-1.963	5.463
-0.168	2.318	-1.667	4.667	-0.019	0.269	-0.228	1.678	-0.065	1.765	-1.951	5.451
-0.169	2.319	-1.646	4.646	-0.019	0.269	-0.225	1.675	-0.065	1.715	-1.936	5.436
-0.169	2.319	-1.669	4.669	-0.019	0.619	-0.227	1.677	-0.065	1.515	-1.919	5.419
-0.168	2.318	-1.648	4.848	-0.019	0.619	-0.225	1.275	-0.064	1.514	-1.903	5.403
-0.167	2.317	-1.615	4.015	-0.019	0.619	-0.223	1.273	-0.064	1.514	-1.873	4.223
-0.165	2.315	-1.616	4.016	-0.018	0.618	-0.217	1.767	-0.062	1.762	-1.707	4.057
-0.164	2.314	-1.604	4.004	-0.018	0.618	-0.217	1.917	-0.062	1.762	-1.833	4.183
-0.163	2.313	-1.599	4.699	-0.017	0.617	-0.208	1.858	-0.059	1.209	-1.838	4.638
-0.161	2.311	-1.591	4.691	-0.018	0.618	-0.211	1.811	-0.058	1.058	-1.781	4.581
-0.161	2.311	-1.565	4.765	-0.017	0.617	-0.205	1.755	-0.057	1.157	-1.817	4.617
-0.157	2.307	-1.558	4.758	-0.017	0.617	-0.205	1.705	-0.057	1.157	-1.805	4.605

-0.156	2.306	-1.540	4.740	-0.017	0.617	-0.205	1.555	-0.056	1.156	-1.801	4.601
-0.156	2.306	-1.540	4.740	-0.017	0.617	-0.204	1.554	-0.054	1.004	-1.774	4.574
-0.155	2.305	-1.514	5.214	-0.017	0.617	-0.204	1.554	-0.054	1.004	-1.724	4.524
-0.154	2.154	-1.440	5.140	-0.017	0.617	-0.192	1.692	-0.054	1.004	-1.467	4.267
-0.151	2.151	-1.462	5.462	-0.016	0.616	-0.176	1.676	-0.053	0.903	-1.482	4.282
-0.150	1.900	-1.462	4.962	-0.016	0.616	-0.191	1.941	-0.053	0.853	-1.696	4.496
-0.149	1.899	-1.431	4.931	-0.016	0.616	-0.191	1.941	-0.052	0.952	-1.625	4.425
-0.148	1.898	-1.396	4.896	-0.016	0.516	-0.182	1.932	-0.052	0.952	-1.665	4.465
-0.145	1.895	-1.415	4.915	-0.015	0.515	-0.188	2.038	-0.052	1.052	-1.655	4.455
-0.144	1.894	-1.416	4.916	-0.015	0.515	-0.182	1.832	-0.052	1.052	-1.629	4.429
-0.143	1.893	-1.400	4.900	-0.015	0.515	-0.161	1.611	-0.051	1.101	-1.593	4.393
-0.142	1.892	-1.388	4.888	-0.015	0.415	-0.177	1.627	-0.050	1.100	-1.602	4.402
-0.136	1.536	-1.367	4.867	-0.015	0.415	-0.169	1.619	-0.050	1.100	-1.590	4.390
-0.134	1.534	-1.358	4.858	-0.015	0.365	-0.171	1.821	-0.049	1.199	-1.575	4.375
-0.134	1.384	-1.345	3.945	-0.015	0.365	-0.158	1.358	-0.049	1.299	-1.541	4.341
-0.135	1.385	-1.330	3.930	-0.014	0.364	-0.167	1.617	-0.049	1.099	-1.480	4.280
-0.135	1.385	-1.324	4.024			-0.166	1.616	-0.047	0.997	-1.414	4.214
-0.134	1.884	-1.279	3.979			-0.163	1.663	-0.046	0.996	-1.288	4.088
-0.131	1.881	-1.282	4.282			-0.154	1.904	-0.046	1.246	-1.430	4.830
-0.130	1.880	-1.272	4.072							-1.432	4.832
-0.128	1.878	-1.250	3.600							-1.415	4.815
-0.127	1.877	-1.167	2.967								
-0.125	1.875	-1.185	3.085								
-0.120	1.670	-1.187	3.187								
-0.120	1.670										
-0.121	2.571										

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